

K -Theory of Azumaya Algebras

A thesis for the degree of Doctor of Philosophy

submitted to

the School of Mathematics and Physics

of

Queen's University Belfast.

Judith Ruth Millar MSci

September 2010

Acknowledgements

I would like to express my gratitude to my supervisor Dr. Roozbeh Hazrat for his advice, encouragement, patience and the many hours of his time over the last four years.

I would also like to thank the members of the Department of Pure Mathematics, and especially the postgraduate students I have met here at Queen's.

I acknowledge the financial assistance I received from the Department of Education and Learning.

Finally, I would like to thank my family for their support throughout my university life.

Contents

Notation	3
0 Introduction	5
1 Azumaya Algebras	12
1.1 Faithfully projective modules	12
1.2 Separable algebras over commutative rings	18
1.3 Other definitions of separability	24
1.4 Azumaya algebras	25
1.5 Further characterisations of Azumaya algebras	29
1.6 The development of the theory of Azumaya algebras	35
2 Algebraic K-Theory	37
2.1 Lower K -groups	37
2.2 Lower K -groups of central simple algebras	44
2.3 Higher K -Theory	51
3 K-Theory of Azumaya Algebras	55
3.1 \mathcal{D} -functors	56
3.2 Homology of Azumaya algebras	61
4 Graded Azumaya Algebras	63
4.1 Graded rings	64
4.2 Graded modules	68

4.3	Graded central simple algebras	76
4.4	Graded matrix rings	80
4.5	Graded projective modules	93
5	Graded K-Theory of Azumaya Algebras	99
5.1	Graded K_0	100
5.2	Graded K_0 of strongly graded rings	105
5.3	Examples	109
5.4	Graded \mathcal{D} -functors	112
6	Additive Commutators	119
6.1	Homogeneous additive commutators	120
6.2	Graded splitting fields	125
6.3	Some results in the non-graded setting	131
6.4	Quotient division rings	133
	Bibliography	138
	Index	145

Notation

Throughout all rings are assumed to be associative and any ring R has a multiplicative identity element 1_R . A subring S of R contains the identity element of R . We assume that a ring homomorphism $R \rightarrow R'$ takes the identity of R to the identity of R' .

Modules over a ring R are assumed to be left R -modules unless otherwise stated. For an R -module M , we assume that $1_R m = m$ for every $m \in M$. For a homomorphism between R -modules, we will use the terms R -module homomorphism and R -linear homomorphism interchangeably.

For a field F , a division algebra D over F is defined to be a division ring with centre F such that $[D : F] < \infty$.

Some symbols used

For a ring R ,

R^*	the group of units of R ; that is, the elements of R which have a multiplicative inverse;
$R\text{-}\mathcal{M}\mathcal{o}\mathcal{d}$	the category of R -modules with R -module homomorphisms between them;
$\text{Pr}(R)$	the subcategory of $R\text{-}\mathcal{M}\mathcal{o}\mathcal{d}$ consisting of finitely generated projective R -modules.

For a multiplicative group G and $x, y \in G$,

$[x, y]$	the commutator $xyx^{-1}y^{-1}$;
$[G, G] = G'$	the <i>commutator subgroup</i> of G ; that is, the (normal) subgroup of G generated by the commutators;
G/G'	the <i>abelianisation</i> of G .

Chapter 0

Introduction

For a division algebra D finite dimensional over its centre F , consider the multiplicative group $D^* = D \setminus 0$. The structure of subgroups of D^* is not known in general. In 1953, Herstein [35] proved that if D is of characteristic $p \neq 0$, then every finite subgroup of D^* is cyclic. This is an easy result in the setting of fields (see [37, Thm. V.5.3]), so the finite subgroups of such division algebras behave in a similar way to those of fields. We give an example here to show that this result doesn't necessarily hold for division algebras of characteristic zero. Hamilton's quaternions \mathbb{H} form a division algebra of characteristic zero, but the subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ is a finite subgroup of \mathbb{H}^* which is not abelian, so not cyclic.

In 1955, Amitsur classified all finite subgroups of D^* in his influential paper [1]. Since then, subgroups of the group D^* have been studied by a number of people (see for example [23, 25, 42, 46]). Herstein [36] showed that a non-central element in a division algebra has infinitely many conjugates. Since normal subgroups are invariant under conjugation, this shows that non-central normal subgroups are “big” in D^* . Note that if D is a non-commutative division algebra, then non-trivial non-central normal subgroups exist in D^* . For example D' , the subgroup of D^* generated by the multiplicative commutators, is non-trivial and is a non-central normal subgroup of D^* .

So as for normal subgroups, we could ask: how large are maximal subgroups of

D^* ? But it remains as an open question whether maximal subgroups even exist in D^* . The existence of maximal subgroups of D^* is connected with the non-triviality of K -group $\mathrm{CK}_1(D)$. For a division algebra D with centre F , we note that $\mathrm{CK}_1(D) \cong D^*/(F^*D')$. The group $\mathrm{CK}_1(D)$ is related to algebraic K -theory; more specifically, to the functor K_1 . Before discussing the group, we indicate how algebraic K -theory has developed.

Algebraic K -theory defines a sequence of functors K_i from the category of rings to the category of abelian groups. For the lower K -groups, the functor K_0 was introduced in the mid-1950s by Grothendieck, and the functor K_1 was developed in the 1960s by Bass. Many attempts were made to extend these functors to cover all K_i for $i \geq 0$. Milnor defined the functor K_2 in the 1960s, but it was not clear how to construct the higher K -functors. The functor K_2 is defined in such a way that there is an exact sequence linking it with K_0 and K_1 (see [58, Thm. 4.3.1]). The “correct” definition of the higher K -functors was required not only to provide such an exact sequence connecting the functors, but also to cover the given definitions of K_0 and K_1 . Then in 1974, Quillen gave two different constructions of the higher K -functors, which are equivalent for rings and which satisfy these expected properties (see [58, Ch. 5]).

It is straightforward to describe the lower K -groups concretely. The group K_0 can be considered as the group completion of the monoid of isomorphism classes of finitely generated projective modules, and K_1 is the abelianisation of the infinite general linear group (see Chapter 2 for the details). The higher K -groups are considerably more difficult to compute. They are, however, functorial in construction.

Returning to the setting of division algebras, consider a central simple algebra A . Then by the Artin-Wedderburn Theorem, A is isomorphic to a matrix $M_n(D)$ over a division algebra D . Let F be the centre of D . Since each K_i , $i \geq 0$, is a functor from the category of rings to the category of abelian groups, the inclusion map $F \rightarrow A$ induces a map $K_i(F) \rightarrow K_i(A)$. Let $\mathrm{ZK}_i(A)$ denote the kernel of this

map and $\mathrm{CK}_i(A)$ denote the cokernel. This gives an exact sequence

$$1 \rightarrow \mathrm{ZK}_i(A) \rightarrow K_i(F) \rightarrow K_i(A) \rightarrow \mathrm{CK}_i(A) \rightarrow 1. \quad (1)$$

So the group $\mathrm{CK}_1(A)$ is defined to be $\mathrm{coker}(K_1(F) \rightarrow K_1(A))$. Then it can be shown that $\mathrm{CK}_1(A)$ is isomorphic to $D^*/F^{*n}D'$, and it is a bounded torsion abelian group (see Section 2.2 for the details).

For a division algebra D , if $\mathrm{CK}_1(M_n(D))$ is not the trivial group for some $n \in \mathbb{N}$, then D^* has a normal maximal subgroup (see [33, Section 2]). It has been conjectured that if $\mathrm{CK}_1(D)$ is trivial, then D is a quaternion algebra (see [30, p. 408]). One of the most significant results in this direction proves that if D is a division algebra with centre F such that D^* has no maximal subgroups, then D and F satisfy a number of conditions (see [33, Thm. 1]). This result ensures that certain division algebras, for example division algebras of degree 2^n or 3^n for $n \geq 1$, have maximal subgroups.

We note that CK_1 can be considered as a functor from the category of central simple algebras over a fixed field to the category of abelian groups (see Section 2.2). In fact, this can be generalised to cover commutative rings. Central simple algebras over fields are generalised by Azumaya algebras over commutative rings. Azumaya algebras were originally defined as “proper maximally central algebras” by Azumaya in his 1951 paper [3]. We outline in Section 1.6 how the definition has developed since then. An Azumaya algebra A over a commutative ring R can be defined as an R -algebra A such that A is finitely generated as an R -module and $A/\mathfrak{m}A$ is a central simple R/\mathfrak{m} -algebra for all $\mathfrak{m} \in \mathrm{Max}(R)$ (see Theorem 1.5.3 for some equivalent definitions).

Then CK_1 can also be considered as a functor from the category of Azumaya algebras over a fixed commutative ring to the category of abelian groups (see page 58). Related to this, various CK_1 -like functors on the categories of central simple algebras and Azumaya algebras have been investigated in [27, 28, 45]. In these papers various abstract functors have been defined, which have similar properties to the functor CK_1 . For example, in [28], the functor defined there is used to show that the K -

theory of an Azumaya algebra over a local ring is almost the same as the K -theory of the base ring.

Along the same lines, in Chapter 3 we define an abstract functor, called a \mathcal{D} -functor, which also has similar properties to CK_1 . We show that the range of a \mathcal{D} -functor is a bounded torsion abelian group, and that CK_i and ZK_i , for $i \geq 0$, are \mathcal{D} -functors. By combining these results with (1), we show in Theorem 3.1.5 that if A is an Azumaya algebra free over its centre R of rank n , then

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]$$

for any $i \geq 0$. This allows us to extend the results of [28] to cover Azumaya algebras over semi-local rings (see Corollary 3.1.7).

Thus far, we have been considering division algebras. Recently Tignol and Wadsworth [67, 66] have studied division algebras equipped with a valuation. Valuations are more common on fields than on division algebras. However they noted that a number of division algebras are equipped with a valuation, and the valuation structure on the division algebra contains a significant amount of information about the division algebra.

A division algebra D equipped with a valuation gives rise to an associated graded division algebra $\mathrm{gr}(D)$. These graded division algebras have been studied in [6, 34, 38, 39, 67]. In these papers, as they are considering graded division algebras associated to division algebras with valuations, their grade groups are totally ordered abelian groups. It was noted in [39] that it is relatively easier to work with graded division algebras, and that not much information is lost in passing between the graded and non-graded settings.

We show in Theorem 4.3.3 that a graded central simple algebra (so, in particular, a graded division algebra) with an abelian grade group is an Azumaya algebra, and therefore the results of Chapter 3 also hold in this setting. But in the graded setting, we can also consider graded finitely generated projective modules over a given graded ring. We define the graded K -theory of a graded ring R to be $K_i^{\mathrm{gr}}(R) = K_i(\mathcal{P}\mathrm{gr}(R))$,

where $\mathcal{P}\text{gr}(R)$ is the category of graded finitely generated projective R -modules.

However, considering graded K -theory of graded Azumaya algebras, Example 5.3.2 gives a graded Azumaya algebra such that its graded K -theory is not isomorphic to the graded K -theory of its centre. In this example, we take the real quaternion algebra \mathbb{H} . Then we show \mathbb{H} is a graded Azumaya algebra over \mathbb{R} , with $K_0^{\text{gr}}(\mathbb{H}) \otimes \mathbb{Z}[1/n] \cong \mathbb{Z} \otimes \mathbb{Z}[1/n]$ and $K_0^{\text{gr}}(Z(\mathbb{H})) \otimes \mathbb{Z}[1/n] \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \otimes \mathbb{Z}[1/n]$, so they are not isomorphic. Thus the results of Chapter 3 do not follow immediately in the setting of graded K -theory.

But for a graded Azumaya algebra subject to certain conditions, we show in Theorem 5.4.4 that its graded K -theory is almost the same as the graded K -theory of its centre. More precisely, for a commutative graded ring R , we let $\Gamma_{M_n(R)}^*$ denote the elements $(d) \in \Gamma^n$ such that $\text{GL}_n(R)[d] \neq \emptyset$, where $\text{GL}_n(R)[d]$ are invertible $n \times n$ matrices with “shifting” (see page 71). We show that if A is a graded Azumaya algebra which is graded free over its centre R of rank n , such that A has a homogeneous basis with degrees $(\delta_1, \dots, \delta_n)$ in $\Gamma_{M_n(R)}^*$, then for any $i \geq 0$,

$$K_i^{\text{gr}}(A) \otimes \mathbb{Z}[1/n] \cong K_i^{\text{gr}}(R) \otimes \mathbb{Z}[1/n].$$

Another K -group which has been studied in the setting of division algebras is the reduced Whitehead group SK_1 (see for example [53]). For a division algebra D , the group $\text{SK}_1(D)$ is defined to be $D^{(1)}/D'$ where $D^{(1)}$ is the kernel of the reduced norm and D' is the group generated by the multiplicative commutators of D . For a graded division algebra D , it has been shown that $\text{SK}_1(QD) \cong \text{SK}_1(D)$, where QD is the quotient division ring of D (see [34, Thm. 5.7]). Related to this, we study additive commutators in the setting of graded division algebras in Chapter 6. We show that for a graded division algebra over its centre F , which is Noetherian as a ring, then

$$\frac{D}{[D, D]} \otimes_F QF \cong \frac{QD}{[QD, QD]}.$$

Summary of the Thesis

In Chapter 1, we combine various results from the literature to show some of the definitions of an Azumaya algebra, their basic properties and the equivalence of some of these definitions. We also outline how the definition has progressed since the work of Azumaya. We note here that Grothendieck [26, §5] also defines an Azumaya algebra on a scheme X with structure sheaf \mathcal{O}_X , but we do not consider that point of view.

In Chapter 2, we begin by recalling the definitions of the lower K -groups K_0 , K_1 and K_2 . We then look at the lower K -groups of central simple algebras, including the functors CK_0 and CK_1 , and some of their properties. We finish the chapter by recalling some properties of the higher K -groups.

In Chapter 3, we define an abstract functor, called a \mathcal{D} -functor, defined on the category of Azumaya algebras over a fixed commutative ring. This allows us to show that the K -theory of an Azumaya algebra free over its centre is almost the same as the K -theory of its centre. We also note that Cortiñas and Weibel [13] have shown a similar result for the Hochschild homology of an Azumaya algebra, which we mention in this chapter.

Chapter 4 introduces graded objects. Often in the literature the grade groups are abelian and totally ordered, so torsion-free. We begin this section by adopting in the graded setting some theorems that we require from the non-graded setting. Some of these results hold for grade groups which are neither abelian nor totally ordered. Though in some cases we require additional conditions on the grade group. We show that for a graded division ring D graded by an arbitrary group, a graded module over D is graded free and has a uniquely defined dimension. For a graded field R and a graded central simple R -algebra A graded by an abelian group, we show that A is a graded Azumaya algebra over R . We also prove a number of results for graded matrix rings graded by arbitrary groups.

We begin Chapter 5 by defining the group K_0 in the setting of graded rings. We show what this group looks like for a trivially graded field and for a strongly graded

ring. For a specific example of a graded Azumaya algebra, we show that its graded K -theory is not the same as its usual K -theory (see Example 5.3.5). Then in a similar way to Chapter 3, we define an abstract functor called a graded \mathcal{D} -functor. This allows us to prove that the graded K -theory of a graded Azumaya algebra (subject to some conditions) is almost the same as the graded K -theory of its centre.

In Chapter 6, we study additive commutators in the setting of graded division algebras. We observe in Section 6.2 that the reduced trace holds in this setting. We then recall the definition of the quotient division algebra, and show in Corollary 6.4.5 how the subgroup generated by homogeneous additive commutators in a graded division algebra relates to that of the quotient division algebra.

Chapter 1

Azumaya Algebras

The concept of an Azumaya algebra over a commutative ring generalises the concept of a central simple algebra over a field. The term Azumaya algebra originates from the work done by Azumaya in his 1951 paper “On maximally central algebras” [3]. The definition has developed since then, and we will outline in Section 1.6 how it has progressed. In Theorem 1.5.3, we state a number of equivalent reformulations of this definition.

This chapter is organised as follows. We begin this chapter by recalling the various definitions of the term “faithfully projective”, which are required for the definition of an Azumaya algebra (see Definition 1.4.1). In Sections 1.2 and 1.3 we discuss separable algebras, which can also be used to define Azumaya algebras. The definition of an Azumaya algebra is stated in Section 1.4, along with some examples and properties, and in Section 1.5 we show some additional characterisations of Azumaya algebras. We conclude this chapter by summarising some of the key progressions in the development of the theory of Azumaya algebras.

1.1 Faithfully projective modules

Let R be a (possibly non-commutative) ring. Consider a covariant additive functor T from the category of (left or right) R -modules to some category of modules. We

say that T is an *exact functor* if, whenever $L \rightarrow M \rightarrow N$ is an exact sequence of R -modules, $T(L) \rightarrow T(M) \rightarrow T(N)$ is exact. Further T is defined to be a *faithfully exact functor* if the sequence $T(L) \rightarrow T(M) \rightarrow T(N)$ is exact if and only if the sequence $L \rightarrow M \rightarrow N$ is exact.

We recall that an R -module M is called *faithful* if $rM = 0$ implies $r = 0$ or, equivalently, if its annihilator $\text{Ann}(M) = \{x \in R : xm = 0 \text{ for all } m \in M\}$ is zero. An R -module M is called a *flat module* if the functor $-\otimes_R M$ is an exact functor from the category of right R -modules to the category of abelian groups. An R -module P is called a *projective module* if the functor $\text{Hom}_R(P, -)$ is an exact functor from the category of left R -modules to the category of abelian groups. This is equivalent to saying that P is a direct summand of a free R -module. If P is a projective R -module which is finitely generated by n elements, then P is a direct summand of R^n . See Magurn [48, Ch. 2] for results involving projective modules.

The following results on faithfully exact functors are from Ishikawa [40, p. 30–33].

Theorem 1.1.1. *Let T be an exact functor from the category of left (resp. right) R -modules to some category of modules. Then the following are equivalent:*

1. T is a faithfully exact functor.
2. $T(A) \neq 0$ for every non zero left (resp. right) R -module A .
3. $T(\phi) \neq 0$ for every non zero R -linear homomorphism ϕ .
4. $T(R/I) \neq 0$ for every proper left (resp. right) ideal I of R .
5. $T(R/\mathfrak{m}) \neq 0$ for every maximal left (resp. right) ideal \mathfrak{m} of R .

We will consider the left version of this theorem in the proof below. The right version follows analogously.

Proof. (1) \Rightarrow (2): Let $T(A) = 0$ for an R -module A . Then since $T(0)$ is the zero module, $T(0) \rightarrow T(A) \rightarrow T(0)$ is exact. By (1), this implies that $0 \rightarrow A \rightarrow 0$ is

exact, proving $A = 0$.

(2) \Rightarrow (3): Let $\phi : X \rightarrow Y$ be an R -linear homomorphism. Then we have exact sequences $X \xrightarrow{\phi'} \text{Im}(\phi) \rightarrow 0$ and $0 \rightarrow \text{Im}(\phi) \xrightarrow{i} Y$, where i is the inclusion map. Since T is an exact functor, we get the following commutative diagram with its row and column exact:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 T(X) & \xrightarrow{T(\phi')} & T(\text{Im}(\phi)) & \longrightarrow & 0 \\
 & \searrow T(\phi) & \downarrow T(i) & & \\
 & & T(Y) & &
 \end{array}$$

If $T(\phi) = 0$, then $T(i) \circ T(\phi') = 0$. This implies that $\text{Im}(T(\phi')) \subseteq \ker(T(i)) = 0$, so $T(\phi') = 0$ and, since $T(\phi')$ is surjective, $T(\text{Im}(\phi)) = 0$. By condition (2), $\text{Im}(\phi) = 0$, so $\phi = 0$.

(3) \Rightarrow (1): Let $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ be exact. Since $T(g \circ f) = T(g) \circ T(f) = 0$, by condition (3), $g \circ f = 0$ and $\text{Im}(f) \subseteq \ker(g)$. We have exact sequences $0 \rightarrow \ker(g) \xrightarrow{j} B \xrightarrow{g} C$, $A \xrightarrow{f'} \text{Im}(f) \rightarrow 0$ and $0 \rightarrow \text{Im}(f) \xrightarrow{i} \ker(g) \xrightarrow{p} \ker(g)/\text{Im}(f) \rightarrow 0$, where i and j are inclusion maps. Since T is an exact functor, we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 0 & \longleftarrow & T(\text{Im}(f)) & \xleftarrow{T(f')} & T(A) & & \\
 & & \downarrow T(i) & & \downarrow T(f) & & \\
 0 & \longrightarrow & T(\ker(g)) & \xrightarrow{T(j)} & T(B) & \xrightarrow{T(g)} & T(C) \\
 & & \downarrow T(p) & & & & \\
 & & T(\ker(g)/\text{Im}(f)) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

For $x \in T(\ker(g))$, $(T(g) \circ T(j))(x) = 0$, so $T(j)(x) \in \ker(T(g)) = \text{Im}(T(f))$.

So there is an element $y \in T(A)$ such that $T(f)(y) = T(j)(x)$. Hence $T(j)(x) = T(f)(y) = T(j) \circ T(i) \circ T(f')(y)$, so that $x = (T(i) \circ T(f'))(y)$, since $T(j)$ is injective. This shows that $T(i)$ is surjective, and hence $T(\text{Im}(f)) \cong T(\ker(g))$, which means that $T(p) = 0$. By condition (3), this implies $p = 0$, so $\text{Im}(f) = \ker(g)$, proving $A \xrightarrow{f} B \xrightarrow{g} C$ is exact.

(2) \Rightarrow (4) and (4) \Rightarrow (5) are trivial.

(5) \Rightarrow (2): Let $T(A) = 0$. Let $a \in A$ and let Ra be the left R -submodule of A generated by a . Since $0 \rightarrow Ra \rightarrow A$ is exact and T is exact, $T(Ra) = 0$. Let $\mathcal{L}(a) = \{r \in R : ra = 0\}$, which is a left ideal of R . If $\mathcal{L}(a) \neq R$, then there is a maximal ideal \mathfrak{m} of R containing $\mathcal{L}(a)$. We have an exact sequence $R/\mathcal{L}(a) \rightarrow R/\mathfrak{m} \rightarrow 0$. Since $R \rightarrow Ra$ is surjective, $Ra \cong R/\mathcal{L}(a)$ by the First Isomorphism Theorem, and we have an exact sequence $0 = T(Ra) \cong T(R/\mathcal{L}(a)) \rightarrow T(R/\mathfrak{m}) \rightarrow 0$. This implies $T(R/\mathfrak{m}) = 0$, contradicting (5). So $\mathcal{L}(a) = R$, which implies $a = 0$ and therefore $A = 0$. \square

For fixed left R -modules P and M , the functors $T(-) = \text{Hom}_R(P, -)$ and $U(-) = - \otimes_R M$ are covariant functors defined on the category of left R -modules and right R -modules, respectively.

Definition 1.1.2. An R -module P is said to be *faithfully projective* if $T(-) = \text{Hom}_R(P, -)$ is a faithfully exact functor and an R -module M is said to be *faithfully flat* if $U(-) = - \otimes_R M$ is a faithfully exact functor.

By applying Theorem 1.1.1 to the functors U and T respectively, we get the following theorems.

Theorem 1.1.3. *Let M be a flat left R -module. Then the following are equivalent:*

1. M is faithfully flat.
2. $A \otimes_R M \neq 0$ for every non zero right R -module A .
3. $\phi \otimes_R \text{id}_M \neq 0$ for every non zero right R -linear homomorphism ϕ .

4. $IM \neq M$ for every proper right ideal I of R .

5. $mM \neq M$ for every maximal right ideal m of R .

Proof. Follows immediately from Theorem 1.1.1. Note that in part (4), $R/I \otimes_R M \cong M/IM$, and $M/IM = 0$ if and only if $M = IM$. \square

Theorem 1.1.4. *Let P be a projective left R -module. Then the following are equivalent:*

1. P is faithfully projective.

2. $\text{Hom}_R(P, A) \neq 0$ for every non zero left R -module A .

3. $\text{Hom}_R(P, \phi) \neq 0$ for every non zero left R -linear homomorphism ϕ .

4. $\text{Hom}_R(P, R/I) \neq 0$ for every proper left ideal I of R .

5. $\text{Hom}_R(P, R/m) \neq 0$ for every maximal left ideal m of R .

Proof. Follows immediately from Theorem 1.1.1. In part (3), if $\phi \in \text{Hom}_R(X, Y)$, then $T(\phi) = \text{Hom}_R(P, \phi) : \text{Hom}_R(P, X) \rightarrow \text{Hom}_R(P, Y); \psi \mapsto \phi \circ \psi$. \square

Proposition 1.1.5. *If an R -module P is faithfully projective, then P is projective and faithfully flat. Further, when the ring R is commutative the converse holds.*

Proof. If an R -module P is faithfully projective, it is projective, and therefore also flat (see [43, Prop. 4.3]). Using Theorem 1.1.3(2), we assume $A \otimes_R P = 0$ and need to prove that $A = 0$. Then by [7, §II.4.1, Prop. 1],

$$\text{Hom}_R(P, \text{Hom}_{\mathbb{Z}}(A, A)) \cong \text{Hom}_{\mathbb{Z}}(A \otimes_R P, A) = \text{Hom}_{\mathbb{Z}}(0, A) = 0.$$

Since P is faithfully projective, by Theorem 1.1.4(2), $\text{Hom}_{\mathbb{Z}}(A, A) = 0$, so $A = 0$.

Conversely, let R be commutative and P be faithfully flat and projective. By Theorem 1.1.3(5), for any maximal ideal m of R , we have $P/mP \neq 0$. Since $R \rightarrow$

R/\mathfrak{m} is a surjective ring homomorphism, R/\mathfrak{m} -linear maps can be considered as R -linear maps and we have $\mathrm{Hom}_R(P/\mathfrak{m}P, R/\mathfrak{m}) = \mathrm{Hom}_{R/\mathfrak{m}}(P/\mathfrak{m}P, R/\mathfrak{m})$. Since R/\mathfrak{m} is a field and $P/\mathfrak{m}P \neq 0$, its dual module $\mathrm{Hom}_{R/\mathfrak{m}}(P/\mathfrak{m}P, R/\mathfrak{m})$ is also non-zero, using

$$\dim_{R/\mathfrak{m}}(\mathrm{Hom}_{R/\mathfrak{m}}(P/\mathfrak{m}P, R/\mathfrak{m})) \geq \dim_{R/\mathfrak{m}}(P/\mathfrak{m}P),$$

from [37, p. 204, Remarks]. We have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(P/\mathfrak{m}P, R/\mathfrak{m}) \longrightarrow \mathrm{Hom}_R(P, R/\mathfrak{m}),$$

so $\mathrm{Hom}_R(P, R/\mathfrak{m}) \neq 0$, since $\mathrm{Hom}_R(P/\mathfrak{m}P, R/\mathfrak{m}) \neq 0$. By Theorem 1.1.4(5), P is faithfully projective. \square

In the following proposition, we show that the definition of a faithfully projective R -module (Definition 1.1.2) can be expressed in a number of different ways, which are equivalent to the definition given above provided R is a commutative ring. The second definition is from [4, p. 39] or [22, p. 186], and the third from [41, p. 52].

Proposition 1.1.6. *Let R be a commutative ring, and let P be an R -module. Then the following are equivalent:*

1. P is faithfully projective;
2. P is finitely generated, projective and faithful as an R -module;
3. P is projective over R and $P \otimes_R N = 0$ implies $N = 0$ for any left R -module N .

Proof. (1) \Leftrightarrow (3): This follows from Proposition 1.1.5 and Theorem 1.1.3(2).

(1) \Leftrightarrow (2): See Bass [4, Cor. II.5.10]. \square

We show below how faithfully flat modules are related to modules which are faithful and flat.

Proposition 1.1.7. *Let R be a ring. A faithfully flat left R -module M is both faithful and flat.*

Proof. See Lam [43, Prop. 4.73], with minor alterations for left modules. \square

In general the converse does not hold. For example, if $R = \mathbb{Z}$, then the module \mathbb{Q} is faithful and flat, since $\mathbb{Q} = (\mathbb{Z} \setminus 0)^{-1}\mathbb{Z}$ is a localisation of \mathbb{Z} , and we know localisations are flat (see [48, Prop. 6.56]). But by Theorem 1.1.3, \mathbb{Q} is not faithfully flat over \mathbb{Z} , since for the ideal $2\mathbb{Z}$ of \mathbb{Z} , $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) = 0$. Even a faithful and projective R -module M is not necessarily faithfully flat over R . For example, let R be the direct product $\mathbb{Z} \times \mathbb{Z} \times \cdots$, and let M be the ideal $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ in R . Then M is faithful as a left R -module, and it is projective (see [43, Eg. 2.12C]). But we have $M^2 = M$, so for any maximal ideal \mathfrak{m} of R containing M , we have $M\mathfrak{m} = M$. So by Theorem 1.1.3, M is not faithfully flat.

1.2 Separable algebras over commutative rings

In this section, we let R denote a commutative ring. Let A be an R -algebra, and let $A^e = A \otimes_R A^{\text{op}}$ be the enveloping algebra of A , where A^{op} denotes the opposite algebra of A . Then the R -algebra A^e has a left action on A induced by:

$$(a \otimes b)x := axb \quad \text{for } a, x \in A, b \in A^{\text{op}},$$

which is denoted by $(a \otimes b) * x$. Any A -bimodule M can also be viewed as a left A^e -module. We set

$$M^A = \{m \in M : ma = am \text{ for all } a \in A\}.$$

There is an A^e -linear map $\mu : A^e \rightarrow A$; $a \otimes b \mapsto (a \otimes b) * 1 = ab$, extended linearly, and we let J denote the kernel of μ .

Definition 1.2.1. An R -algebra A is said to be *separable* over R if A is projective as a left A^e -module.

The following two theorems show some equivalent characterisations of separability.

Theorem 1.2.2. *Let A be an R -algebra. The following are equivalent:*

1. A is separable.
2. The exact sequence of left A^e -modules

$$0 \longrightarrow J \longrightarrow A^e \xrightarrow{\mu} A \longrightarrow 0$$

splits.

3. The functor $(-)^A : A^e\text{-Mod} \rightarrow R\text{-Mod}$ is exact.
4. There is an element $e \in A^e$ such that $e * 1 = 1$ and $Je = 0$.
5. There is an element $e \in A^e$ such that $e * 1 = 1$ and $(a \otimes 1)e = (1 \otimes a)e$ for all $a \in A$.

Such an element e as in Theorem 1.2.2 is an idempotent, called a *separability idempotent* for A , since $e^2 - e = (e - 1 \otimes 1)e \in Je = 0$.

Proof. [41, Lemma III.5.1.2], [17, Prop. II.1.1]

(1) \Leftrightarrow (2): The forward direction follows immediately from the definition of a projective module. For the converse, using known results involving projective modules (see [48, Cor. 2.16]), (2) implies $A^e \cong J \oplus A$, so A is projective.

(1) \Leftrightarrow (3): For all A -bimodules M , the natural map

$$\begin{aligned} \rho_M : \text{Hom}_{A^e}(A, M) &\longrightarrow M^A \\ f &\longmapsto f(1) \end{aligned}$$

is an isomorphism of R -modules, with the inverse being

$$\begin{aligned}\rho_M^{-1} : M^A &\longrightarrow \operatorname{Hom}_{A^e}(A, M) \\ x &\longmapsto \mathcal{R}_x : A \rightarrow M \\ &a \mapsto ax.\end{aligned}$$

Since A is separable if and only if $\operatorname{Hom}_{A^e}(A, -)$ is an exact functor, this proves the equivalence of (1) and (3).

(2) \Rightarrow (4): Let $\gamma : A \rightarrow A^e$ be an A^e -module homomorphism such that $\mu \circ \gamma = \operatorname{id}_A$. Let $e = \gamma(1)$, so that $1 = \mu(e) = e * 1$. To show that $Je = 0$, let $a \in J$. Then as γ is A^e -linear, $ae = a\gamma(1) = \gamma(a * 1) = 0$, since we have $a * 1 = \mu(a) = 0$, proving $Je = 0$, as required for (4).

(4) \Rightarrow (5): From (4), we have an element $e \in A^e$ such that $e * 1 = 1$ and $Je = 0$. Let $a \in A$ be arbitrary. Then $\mu(1 \otimes a - a \otimes 1) = 0$, so $1 \otimes a - a \otimes 1 \in J$. Hence $(1 \otimes a - a \otimes 1)e = 0$; that is, $(1 \otimes a)e = (a \otimes 1)e$, proving (5).

(5) \Rightarrow (2): If e is an element of A^e satisfying the conditions in (5), we can define a map γ by $\gamma : A \rightarrow A^e; a \mapsto (1 \otimes a)e$. Using the assumption that $(a \otimes 1)e = (1 \otimes a)e$ for all $a \in A$, we can show that γ is an A^e -module homomorphism. It is a right inverse of μ since, for $a \in A$, writing $e = \sum x_i \otimes y_i$ gives

$$\begin{aligned}\mu \circ \gamma(a) &= \mu((1 \otimes a)e) \\ &= \mu\left(\sum (x_i \otimes y_i a)\right) \\ &= \left(\sum x_i y_i\right) a \\ &= 1.a = \operatorname{id}_A(a),\end{aligned}$$

completing the proof. □

Theorem 1.2.3. *Let A be an R -algebra which is finitely generated as an R -module. The following are equivalent:*

1. A is separable over R .
2. $A_{\mathfrak{m}}$ is separable over $R_{\mathfrak{m}}$ for all $\mathfrak{m} \in \text{Max}(R)$.
3. $A/\mathfrak{m}A$ is separable over R/\mathfrak{m} for all $\mathfrak{m} \in \text{Max}(R)$.

Proof. See [41, Lemma III.5.1.10]. □

For a free R -module F , we know that F is isomorphic to a direct sum of copies of R as a left R -module; that is, $F \cong \bigoplus_{i \in I} R_i$ for $R_i = R$ (see [37, Thm. IV.2.1]). Let $f_i \in \text{Hom}_R(F, R)$ be the projection of R_i onto R , and let e_i be the element of F with 1 in the i -th position and zeros elsewhere. Then clearly the following results hold:

1. for every $x \in F$, $f_i(x) = 0$ for all but a finite subset of $i \in I$;
2. for every $x \in F$, $\sum_i f_i(x)e_i = x$.

The following lemma shows that we have similar results when we consider projective modules, rather than free modules. Moreover, such properties are sufficient to characterise a projective module.

Lemma 1.2.4 (Dual Basis Lemma). *Let M be an R -module. Then M is projective if and only if there exists $\{m_i\}_{i \in I} \subseteq M$ and $\{f_i\}_{i \in I} \subseteq \text{Hom}_R(M, R)$, for some indexing set I , such that*

1. *for every $m \in M$, $f_i(m) = 0$ for all but a finite subset of $i \in I$; and*
2. *for every $m \in M$, $\sum_{i \in I} f_i(m)m_i = m$.*

Moreover, I can be chosen to be a finite set if and only if M is finitely generated.

The collection $\{f_i, m_i\}$ is called a *dual basis* for M .

Proof. See [17, Lemma I.1.3]. □

The following proposition, from Villamayor, Zelinsky [69, Prop. 1.1] (see also [17, Prop. II.2.1]), shows that an algebra which is separable and projective is finitely generated. This is a somewhat surprising result, as the requirement of being separable and projective does not immediately appear to imply a finitely generated condition. The proof of the proposition uses the Dual Basis Lemma.

Proposition 1.2.5. *Let A be a separable R -algebra which is projective as an R -module. Then A is finitely generated as an R -module.*

Proof. Since A is projective as an R -module, A^{op} is also projective as an R -module. Let $\{f_i, a_i\}$ be a dual basis for A^{op} over R , where $a_i \in A^{\text{op}}$ and $f_i \in \text{Hom}_R(A^{\text{op}}, R)$. Then for every $b \in A^{\text{op}}$, $b = \sum_{i \in I} f_i(b)a_i$ and $f_i(b) = 0$ for all but finitely many $i \in I$ (using Lemma 1.2.4). Since $A \otimes_R R \cong A$, we can identify $A \otimes_R R$ with A and can consider $\text{id}_A \otimes f_i$ as a map from A^e to A . This map is A -linear, and we claim that $\{\text{id}_A \otimes f_i, 1 \otimes a_i\}$ forms a dual basis for A^e as a projective left A -module. Let $a \otimes b \in A^e$ be arbitrary. Since $f_i(b) = 0$ for all but a finite number of subscripts i , we also have $(\text{id}_A \otimes f_i)(a \otimes b) = a \otimes f_i(b) = 0$ for all but a finite number of i . Then

$$\begin{aligned} \sum_{i \in I} (\text{id}_A \otimes f_i)(a \otimes b)(1 \otimes a_i) &= \sum_{i \in I} a \otimes f_i(b)a_i \\ &= a \otimes b. \end{aligned}$$

Extended linearly, this holds for all $u \in A^e$. So $\{\text{id}_A \otimes f_i, 1 \otimes a_i\}$ forms a dual basis for A^e over A .

Let $a \in A^{\text{op}}$ be a fixed arbitrary element. We will show that a can be written as an R -linear combination of a finite subset of A^{op} , where this finite subset is independent of a . Let $e = \sum_j x_j \otimes y_j$ be a separability idempotent for A over R and define $u = (1 \otimes a)e \in A^e$. We have $u * 1 = \sum_j x_j y_j a = a$ and, from Lemma 1.2.4,

$u = \sum_i (\text{id}_A \otimes f_i)((1 \otimes a)e)(1 \otimes a_i)$. Then

$$\begin{aligned} a &= u * 1 = \sum_i \left(((\text{id}_A \otimes f_i)((1 \otimes a)e) \otimes a_i) * 1 \right) \\ &= \sum_i ((\text{id}_A \otimes f_i)((1 \otimes a)e)) \cdot a_i. \end{aligned} \tag{1.1}$$

Using Proposition 1.2.2(5),

$$\begin{aligned} (\text{id}_A \otimes f_i)((1 \otimes a)e) &= (\text{id}_A \otimes f_i)((a \otimes 1)e) \\ &= a(\text{id}_A \otimes f_i)(e) \\ &= (a \otimes 1) * ((\text{id}_A \otimes f_i)(e)). \end{aligned}$$

Since $\{\text{id}_A \otimes f_i, 1 \otimes a_i\}$ forms a dual basis for A^e , $(\text{id}_A \otimes f_i)(e) = 0$ for all but a finite subset of $i \in I$. So the set of subscripts i for which $(\text{id}_A \otimes f_i)((1 \otimes a)e)$ is non-zero is contained in the finite set of subscripts for which $(\text{id}_A \otimes f_i)(e)$ is non-zero, which is independent of a . Then since $(\text{id}_A \otimes f_i)((1 \otimes a)e)$ is non-zero for only finitely many $i \in I$, the sum (1.1) may be taken over a finite set. Again writing $e = \sum_j x_j \otimes y_j$, (1.1) says:

$$\begin{aligned} a &= \sum_{i,j} (x_j \otimes f_i(y_j a)) \cdot a_i \\ &= \sum_{i,j} x_j f_i(y_j a) a_i \\ &= \sum_{i,j} f_i(y_j a) x_j a_i. \end{aligned}$$

So the finite set $\{x_j a_i\}$ generates A^{op} over R , and therefore generates A over R . This completes the proof that A is finitely generated. \square

1.3 Other definitions of separability

In Definition 1.2.1 above, we define separability over a commutative ring R . If R is a field, we also have the classical definition of separability. For a field R , an R -algebra A is said to be *classically separable* if, for every field extension L of R , the Jacobson radical of $A \otimes_R L$ is zero, where the Jacobson radical of $A \otimes_R L$ is the intersection of the maximal left ideals. The following theorem shows the connection between the two definitions of separability when R is a field.

Theorem 1.3.1. *Let R be a field and A be an R -algebra. Then A is separable over R if and only if A is classically separable over R and the dimension of A as a vector space over R is finite.*

Proof. See [17, Thm. II.2.5]. □

There is a further definition of separability for fields. For a field R , an irreducible polynomial $f(x) \in R[x]$ is separable over R if f has no repeated roots in any splitting field. An algebraic field extension A of R is said to be a *separable field extension* of R if, for every $a \in A$, the minimal polynomial of a over R is separable. The theorem below shows that for a finite field extension, this definition agrees with the definition of classical separability given above. The theorem also shows their connection with the definition of a separable algebra given in Definition 1.2.1.

Theorem 1.3.2. *Let R be a field, and let A be a finite field extension of R . Then the following are equivalent:*

1. A is separable as an R -algebra,
2. A is classically separable over R ,
3. A is a separable field extension of R .

Proof. (1) \Leftrightarrow (2): Follows immediately from Theorem 1.3.1.

(2) \Leftrightarrow (3): See [71, Lemma 9.2.8]. □

1.4 Azumaya algebras

Let R be a commutative ring and A be an R -algebra. There is a natural R -algebra homomorphism $\psi_A : A^e \rightarrow \text{End}_R(A)$ defined by $\psi_A(a \otimes b)(x) = axb$, extended linearly. If the context is clear, we will drop the subscript A . We now are ready to define an Azumaya algebra: this is the definition from [22, p. 186] and [41, p. 134].

Definition 1.4.1. An R -algebra A is called an *Azumaya algebra* if the following two conditions hold:

1. A is a faithfully projective R -module.
2. The map $\psi_A : A^e \rightarrow \text{End}_R(A)$ defined above is an isomorphism.

Example 1.4.2. Any finite dimensional central simple algebra A over a field F is an Azumaya algebra. A central simple algebra is free, so it is projective and faithful, and we know $A \otimes_F A^{\text{op}} \cong M_n(F) \cong \text{End}_F(A)$ (see [63, Thm. 8.3.4]).

We will see some further examples of Azumaya algebras on pages 28 and 32.

Proposition 1.4.3. *Let E_1, E_2, F_1, F_2 be R -modules. When one of the ordered pairs $(E_1, E_2), (E_1, F_1), (E_2, F_2)$ consists of finitely generated projective R -modules, the canonical homomorphism*

$$\text{Hom}(E_1, F_1) \otimes \text{Hom}(E_2, F_2) \longrightarrow \text{Hom}(E_1 \otimes E_2, F_1 \otimes F_2)$$

is bijective.

Proof. See [7, §II.4.4, Prop. 4]. □

We observe that if $E_1, \dots, E_n, F_1, \dots, F_m$ are any R -modules and

$$\phi : E_1 \oplus \dots \oplus E_n \longrightarrow F_1 \oplus \dots \oplus F_m$$

is an R -module homomorphism, then ϕ can be represented by a unique matrix

$$\begin{pmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & & \vdots \\ \phi_{m1} & \cdots & \phi_{mn} \end{pmatrix}$$

where $\phi_{ij} \in \text{Hom}_R(E_j, F_i)$. In particular for R -modules M and N there are R -module homomorphisms

$$\begin{array}{ccccc} \text{End}_R(M) & \xrightarrow{i} & \text{End}_R(M \oplus N) & \xrightarrow{j} & \text{End}_R(M) \\ f & \mapsto & \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} & & \\ & & \begin{pmatrix} x & y \\ z & w \end{pmatrix} & \xrightarrow{j} & x \end{array}$$

Note that $j \circ i = \text{id}$, so that i is injective and j is surjective.

The proposition below shows how to construct some examples of Azumaya algebras. The proof is from Farb, Dennis [22, Prop. 8.3].

Proposition 1.4.4. *If P is a faithfully projective R -module, then $\text{End}_R(P)$ is an Azumaya algebra over R .*

Proof. Using Proposition 1.1.6, P is a finitely generated projective R -module, so we can choose an R -module Q with $P \oplus Q \cong R^n$ for some n . Hence $\text{End}_R(P \oplus Q) \cong \text{End}_R(R^n) \cong M_n(R) \cong R^{n^2}$ as R -modules, so $\text{End}_R(P \oplus Q)$ is a free module. We saw above that there are homomorphisms $\text{End}_R(P) \xrightarrow{i} \text{End}_R(P \oplus Q) \xrightarrow{j} \text{End}_R(P)$ with $j \circ i = \text{id}$. Then $\text{End}_R(P)$ is isomorphic to a direct summand of $\text{End}_R(P \oplus Q)$ and therefore $\text{End}_R(P)$ is finitely generated and projective as an R -module. Suppose $r \in R$ annihilates $\text{End}_R(P)$. In particular r annihilates the identity map on P , and so $rp = 0$ for all $p \in P$. But P is a faithful module, so $r = 0$, which shows that $\text{End}_R(P)$ is also a faithful R -module. By Proposition 1.1.6, this shows that $\text{End}_R(P)$ is faithfully projective.

It remains to show the second condition. Consider the following diagram:

$$\begin{array}{ccc}
 \text{End}_R(P) \otimes \text{End}_R(P)^{\text{op}} & \xrightarrow{\psi_P} & \text{End}_R(\text{End}_R(P)) \\
 i' \downarrow \uparrow j' & & i'' \downarrow \uparrow j'' \\
 \text{End}_R(P \oplus Q) \otimes (\text{End}_R(P \oplus Q))^{\text{op}} & \xrightarrow{\psi_{P \oplus Q}} & \text{End}_R(\text{End}_R(P \oplus Q))
 \end{array}$$

where ψ_P and $\psi_{P \oplus Q}$ are defined as in Definition 1.4.1, and the maps i', j', i'' and j'' come from the homomorphisms i and j on page 26. For an element $f \otimes g$ in $\text{End}_R(P) \otimes \text{End}_R(P)^{\text{op}}$, we have

$$\begin{aligned}
 \psi_{P \oplus Q} \circ i'(f \otimes g) &= i'' \circ \psi_P(f \otimes g) : \text{End}_R(P \oplus Q) \longrightarrow \text{End}_R(P \oplus Q) \\
 \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} &\longmapsto \begin{pmatrix} f \circ \alpha_{11} \circ g & 0 \\ 0 & 0 \end{pmatrix}
 \end{aligned}$$

and for $f \otimes g = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \otimes \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \text{End}_R(P \oplus Q) \otimes (\text{End}_R(P \oplus Q))^{\text{op}}$, we have

$$\begin{aligned}
 \psi_P \circ j'(f \otimes g) &= j'' \circ \psi_{P \oplus Q}(f \otimes g) : \text{End}_R(P) \longrightarrow \text{End}_R(P) \\
 \alpha &\longmapsto f_{11} \circ \alpha \circ g_{11}.
 \end{aligned}$$

So the diagram above commutes and we can show that $j'' \circ i'' = \text{id}_{\text{End}(\text{End}(P))}$ and $j' \circ i' = \text{id}_{\text{End}(P) \otimes \text{End}(P)^{\text{op}}}$. Therefore to show that ψ_P is an isomorphism it is sufficient to show that $\psi_{P \oplus Q}$ is an isomorphism.

Let $\{e_1, \dots, e_n\}$ be a basis for the free R -algebra $P \oplus Q \cong R^n$. Let $E_{ij} \in \text{End}_R(R^n)$ be defined by $E_{ij}(e_k) = \delta_{jk}e_i$. We can consider E_{ij} as the $n \times n$ matrix with 1 in the i - j entry, and zeros elsewhere. Then $\{E_{ij} : 1 \leq i, j \leq n\}$ is an R -module basis for $\text{End}_R(R^n)$ and $\{E_{ij} \otimes E_{kl} : 1 \leq i, j, k, l \leq n\}$ is an R -module basis for $\text{End}_R(R^n) \otimes (\text{End}_R(R^n))^{\text{op}}$. By definition of $\psi_{P \oplus Q}$, we have

$$\psi_{P \oplus Q}(E_{ij} \otimes E_{kl})(E_{st}) = E_{ij} \circ E_{st} \circ E_{kl} = \delta_{js}\delta_{tk}E_{il}. \quad (1.2)$$

Using this, we will show that $\psi_{P \oplus Q}$ is an isomorphism. Let $h : \text{End}_R(R^n) \rightarrow \text{End}_R(R^n)$ be an R -module homomorphism. For an arbitrary basis element $E_{xy} \in \text{End}_R(R^n)$, suppose $h(E_{xy}) = \sum_{i,j} r_{ij}^{(x,y)} E_{ij}$, where $r_{ij}^{(x,y)}$ is an element of R indexed by i, j, x, y . Then define a map

$$\begin{aligned} \phi : \text{End}_R(\text{End}_R(R^n)) &\longrightarrow \text{End}_R(R^n) \otimes (\text{End}_R(R^n))^{\text{op}} \\ h &\longmapsto \sum_{x,y} \sum_{i,j} r_{ij}^{(x,y)} E_{ix} \otimes E_{yj}. \end{aligned}$$

Then we can show that ϕ is an R -algebra homomorphism inverse to $\psi_{P \oplus Q}$, completing the proof. \square

Example 1.4.5. For any commutative ring R , $M_n(R)$ is an Azumaya algebra over R . This follows by applying Proposition 1.4.4: since R^n is free and therefore faithfully projective over R , $\text{End}_R(R^n)$ is an Azumaya algebra over R , and we know that $M_n(R) \cong \text{End}_R(R^n)$ as R -algebras.

In Proposition 1.4.6 below we show that the tensor product of two R -Azumaya algebras is again an R -Azumaya algebra. The proof is from Farb, Dennis [22, Prop. 8.4].

Proposition 1.4.6. *If A and B are Azumaya algebras over R , then $A \otimes_R B$ is an Azumaya algebra over R .*

Proof. Since A and B are finitely generated projective R -modules (by Proposition 1.1.6), we can choose R -modules A' and B' with $A \oplus A' \cong R^n$ and $B \oplus B' \cong R^m$. Then $(A \oplus A') \otimes (B \oplus B') \cong R^n \otimes R^m \cong R^{nm}$. So there is an R -module Q with $(A \otimes B) \oplus Q \cong R^{nm}$, proving $A \otimes B$ is finitely generated and projective.

We know that any projective module is flat, so A is flat over R . Since $R \rightarrow B; r \mapsto r \cdot 1_B$ is injective, it follows that

$$\begin{aligned} f : A &\longrightarrow A \otimes_R R \longrightarrow A \otimes_R B \\ a &\longmapsto a \otimes 1_R \longmapsto a \otimes 1_B \end{aligned}$$

is injective. Let $r \in R$ be such that $r \in \text{Ann}(A \otimes B)$. Then $r(a \otimes 1) = 0$ for all $a \in A$, so $0 = r(f(a)) = f(ra)$, and since f is injective this implies that $ra = 0$ for all $a \in A$. Thus $r \in \text{Ann}(A)$ and so $r = 0$, as A is faithful, proving $A \otimes B$ is also faithful. So by Proposition 1.1.6, $A \otimes B$ is faithfully projective.

Let $\psi_{A \otimes B}$ be the homomorphism defined in the definition of an Azumaya algebra. Then the following diagram is commutative:

$$\begin{array}{ccc} (A \otimes A^{\text{op}}) \otimes (B \otimes B^{\text{op}}) & \xrightarrow{\psi_A \otimes \psi_B} & \text{End}_R(A) \otimes \text{End}_R(B) \\ \theta \downarrow & & \downarrow w \\ (A \otimes B) \otimes (A \otimes B)^{\text{op}} & \xrightarrow{\psi_{A \otimes B}} & \text{End}_R(A \otimes B) \end{array}$$

Here ψ_A and ψ_B are isomorphisms since A and B are Azumaya algebras, w is the isomorphism given by Proposition 1.4.3, and the isomorphism θ comes from the commutativity of the tensor product and the fact that $(A \otimes B)^{\text{op}} \cong A^{\text{op}} \otimes B^{\text{op}}$. This shows that $\psi_{A \otimes B}$ is an isomorphism. \square

1.5 Further characterisations of Azumaya algebras

In this section, unless otherwise stated, R denotes a commutative ring and A is an R -algebra. The definition of an Azumaya algebra (Definition 1.4.1) has a number of equivalent reformulations which are shown in Theorem 1.5.3 below. We firstly require some additional definitions.

We say that an R -algebra A is *central* over R if A is faithful as an R -module and the centre of A coincides with the image of R in A . Thus an R -algebra A is central if and only if the ring homomorphism $f : R \rightarrow Z(A)$ is both injective and surjective. A commutative R -algebra S is said to be a *finitely presented algebra* if S is isomorphic to the quotient ring $R[x_1, \dots, x_n]/I$ of a polynomial ring $R[x_1, \dots, x_n]$ by a finitely generated two-sided ideal I . A commutative R -algebra S is called *étale* if S is flat, finitely presented and separable over R .

Let R be a ring, which is not necessarily commutative. Consider a covariant

additive functor T from the category of (left or right) R -modules \mathcal{C} to some category of modules \mathcal{D} . The functor T induces a function

$$T_{X,Y} : \text{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{D}}(T(X), T(Y))$$

for every pair of objects X and Y in \mathcal{C} . The functor T is said to be

- *faithful* if $T_{X,Y}$ is injective;
- *full* if $T_{X,Y}$ is surjective;
- *fully faithful* if $T_{X,Y}$ is bijective;

for each X and Y in \mathcal{C} . An R -module M is called a *generator* if the functor $\text{Hom}_R(M, -)$ is a faithful functor from the category of left R -modules to the category of abelian groups.

There is another definition of a generator module in [17, p. 5], which is shown to be equivalent to the definition above (see Proposition 1.5.1). For a ring R (not necessarily commutative) and any R -module M , consider the subset $\mathcal{T}_R(M)$ of R consisting of elements of the form $\sum_i f_i(m_i)$ where the f_i are from $\text{Hom}_R(M, R)$ and the m_i are from M . Then $\mathcal{T}_R(M)$ is a two-sided ideal of R , called the *trace ideal* of M . Then [17] defines M to be a generator module if $\mathcal{T}_R(M) = R$.

Proposition 1.5.1. *Let R be a (possibly non-commutative) ring. For any R -module M , the functor $\text{Hom}_R(M, -)$ is a faithful functor if and only if $\mathcal{T}_R(M) = R$.*

Proof. See [43, Thm. 18.8]. □

Let R be a ring, which is not necessarily commutative. An R -module M is called a *projective generator* (or *progenerator*) if M is finitely generated, projective and a generator.

Theorem 1.5.2. *Let R be a commutative ring. Then an R -module M is an R -progenerator if and only if M is finitely generated, projective and faithful.*

Proof. See [17, Cor. I.1.10]. □

Theorem 1.5.3. *Let R be a commutative ring and A be an R -algebra. The following are equivalent:*

1. *A is an Azumaya algebra.*
2. *A is central and separable as an R -module.*
3. *A is central over R and A is a generator as an A^e -module.*
4. *The functors*

$$A\text{-}\mathcal{M}\text{od-}A \longrightarrow \mathcal{M}\text{od-}R, M \longmapsto M^A$$

$$\mathcal{M}\text{od-}R \longrightarrow A\text{-}\mathcal{M}\text{od-}A, N \longmapsto N \otimes A$$

are inverse equivalences of categories. Further projective modules correspond to projective modules.

5. *A is a finitely generated R -module and $A/\mathfrak{m}A$ is a central simple R/\mathfrak{m} -algebra for all $\mathfrak{m} \in \text{Max}(R)$.*
6. *There is a faithfully flat étale R -algebra S and a faithfully projective S -module P such that $A \otimes_R S \cong \text{End}_S(P)$. If R is local, S can be taken as finite étale.*

Proof. The proof follows by combining various parts from [17, Thm. II.3.4], [4, Thm. III.4.1] and [41, Thm. III.5.1.1].

(1) \Leftrightarrow (2): See DeMeyer, Ingraham [17, Thm. II.3.4].

(1) \Leftrightarrow (3): See Bass [4, Thm. III.4.1].

(1) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6): See Knus [41, Thm. III.5.1.1]. □

We state another example of an Azumaya algebra, which follows from the theorem above.

Example 1.5.4. Let R be a commutative ring in which 2 is invertible. Define the quaternion algebra Q over R to be the free R -module with basis $\{1, i, j, k\}$ and with multiplication satisfying $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. As for quaternion algebras over fields (see [52, Lemma 1.6]), it follows that Q is a central R -algebra. Then [65, Cor. 4] says that Q is separable over R . Their proof considers the following element of Q :

$$e = \frac{1}{4}(1 \otimes 1 - i \otimes i - j \otimes j - k \otimes k).$$

It is routine to show that e is a separability idempotent for Q , so Q is separable over R . It follows from Theorem 1.5.3(2) that Q is an Azumaya algebra over R .

Remark 1.5.5. If A is an Azumaya algebra over R , then by Theorem 1.5.3, $A/\mathfrak{m}A$ is a central simple R/\mathfrak{m} -algebra for all $\mathfrak{m} \in \text{Max}(R)$. If A is free over R , then

$$[A : R] = [R/\mathfrak{m} \otimes_R A : R/\mathfrak{m}] = [A/\mathfrak{m}A : R/\mathfrak{m}].$$

Since we know the dimension of a central simple algebra is a square number (see [63, Cor. 8.4.9]), the same is true for A .

We also remark that Azumaya algebras are closely related to Polynomial Identity rings, thanks to the Artin-Procesi Theorem (see [61, §1.8], [62, §6.1]). While we do not consider polynomial identity theory here, we mention a result of Braun in the following theorem. In [9], Braun generalises the Artin-Procesi Theorem and, as a consequence of this generalisation, he gives another characterisation of Azumaya algebras in [9, Thm. 4.1], which we state in the theorem below. The proof given below is a direct proof of this characterisation, which is due to Dicks [19]. The notation used in the proof is defined in Section 1.2.

Theorem 1.5.6. *Let A be a central R -algebra. Then A is an Azumaya algebra over R if and only if there is some $e \in A^e$ such that $e * 1 = 1$ and $e * A \subseteq R$.*

Proof. Using Theorem 1.5.3, A is an Azumaya algebra over R if and only if A is separable over R (since we assumed central in the statement of the theorem). Assume

A is an Azumaya algebra over R . By Theorem 1.2.2(5), this implies there exists an idempotent $e \in A^e$ such that $e * 1 = 1$ and $(a \otimes 1)e = (1 \otimes a)e$ for all $a \in A$. Let $a \in A$ be arbitrary. Then $a(e * A) = (a \otimes 1) * (e * A) = ((a \otimes 1)e) * A = ((1 \otimes a)e) * A = (1 \otimes a) * (e * A) = (e * A)a$, proving $e * A \subseteq Z(A) = R$.

Conversely, suppose there is some $e \in A^e$ with $e * 1 = 1$ and $e * A \subseteq R$. We want to show that A is separable over R ; that is, there is an $e \in A^e$ such that $e * 1 = 1$ and $Je = 0$ (using Theorem 1.2.2). We will first show that $A^e e A^e = A^e$, where $A^e e A^e$ is the ideal of A^e generated by e . Let $I = \{a \in A : a \otimes 1 \in A^e e A^e\}$, which is a two-sided ideal of A . If $I = A$, then as $1 \in A$, $1 \otimes 1 \in A^e e A^e$ and so $A^e e A^e = A^e$.

Assume $I \neq A$. Then there is a maximal ideal M of A such that $I \subseteq M \subsetneq A$. We can give A^e two left A^e -module structures as follows:

$$\begin{aligned} A^e \times A^e &\longrightarrow A^e \\ (u, a \otimes b) &\longmapsto u *_1 (a \otimes b) = (u * a) \otimes b \\ (u, a \otimes b) &\longmapsto u *_2 (a \otimes b) = a \otimes (u * b) \end{aligned}$$

It is routine to check that these are well-defined. Then $e *_2 (a \otimes b) = a \otimes (e * b) = a(e * b) \otimes 1$ and we can show that $A^e *_2 (A^e e A^e) \subseteq A^e e A^e$. It follows that every element of $e *_2 (A^e e A^e)$ is of the form $a \otimes 1$ for some $a \in I \subseteq M$.

Let $\overline{A} = A/M$ and $\overline{R} = R/(R \cap M)$. Then $\overline{R} \hookrightarrow \overline{A}$ and $\overline{R} \subseteq Z(\overline{A})$. Let $\overline{A^e} = \overline{A} \otimes_{\overline{R}} \overline{A^e}^{\text{op}}$. Then $\overline{e} * \overline{1} = \overline{1}$ and $\overline{e} * \overline{A} \subseteq \overline{R}$. To show $\overline{R} = Z(\overline{A})$, let $\overline{x} \in Z(\overline{A})$. Then $\overline{x} = \overline{e} * \overline{x}$, which shows $\overline{x} \in \overline{R}$ since we know $\overline{e} * \overline{x} \in \overline{R}$.

Since M is a maximal ideal of A , \overline{A} is simple, and therefore central simple, over \overline{R} . Also $\overline{A^e}^{\text{op}}$ is simple, so $\overline{A^e}$ is simple. Since $\overline{e} * \overline{1} = \overline{1}$, $\overline{e} \neq 0$ and thus $\overline{A^e} \overline{e} \overline{A^e} = \overline{A^e}$, as $\overline{A^e} \overline{e} \overline{A^e}$ is the two-sided ideal generated by \overline{e} . Then $\overline{1} \otimes \overline{1} = \overline{e} *_2 (\overline{1} \otimes \overline{1}) \in \overline{e} *_2 \overline{A^e} \overline{e} \overline{A^e} = \overline{e} *_2 \overline{A^e} \overline{A^e}$. But we observed above that every element of $e *_2 A^e e A^e$ is of the form $a \otimes 1$ for some $a \in M$, so every element of $\overline{e} *_2 \overline{A^e} \overline{A^e}$ is of the form $\overline{a} \otimes \overline{1} = \overline{a} \otimes \overline{1} = 0$, and therefore $\overline{e} *_2 \overline{A^e} \overline{A^e} = 0$. Since $\overline{1} \otimes \overline{1} \in \overline{e} *_2 \overline{A^e} \overline{A^e}$, this is a contradiction, so our assumption that $I \neq A$ was incorrect, proving $A^e e A^e = A^e$.

It remains to prove $Je = 0$. Let $u \in A^e$. We claim that $A^e e A^e *_2 u \subseteq (u *_1 A^e) A^e$. Let $v, w \in A^e$ and let $u = \sum_i a_i \otimes b_i$. Then

$$\begin{aligned} (vew) *_2 \left(\sum_i a_i \otimes b_i \right) &= \sum_i (a_i \otimes (vew *_1 b_i)) \\ &= \sum_i \left(a_i \otimes \left(v *_1 (e *_1 (w *_1 b_i)) \right) \right) \end{aligned}$$

But $e *_1 (w *_1 b_i) \in R$ as $e *_1 A \subseteq R$, so $v *_1 (e *_1 (w *_1 b_i)) = (v *_1 1)(ew *_1 b_i)$. Then letting $ew = \sum_j c_j \otimes d_j$, we have

$$\begin{aligned} (vew) *_2 \left(\sum_i a_i \otimes b_i \right) &= \sum_i a_i (ew *_1 b_i) \otimes (v *_1 1) \\ &= \sum_i \sum_j (a_i c_j b_i d_j) \otimes (v *_1 1) \\ &= \sum_j \left(\left(\sum_i a_i \otimes b_i \right) *_1 (c_j \otimes (v *_1 1)) \right) (d_j \otimes 1), \end{aligned}$$

proving the claim.

Since $*$ defines a left module action on A and since $e *_1 A \subseteq R$, it follows that $Je *_1 A = J *_1 (e *_1 A) \subseteq J *_1 R$. Then it is easy to see that $J *_1 R = 0$, and so $Je *_1 A = 0$. But $Je \subseteq A^e *_2 (Je) = A^e e A^e *_2 (Je) \subseteq (Je *_1 A^e) A^e = 0$, proving $Je = 0$ as required. \square

We recall that for two rings A and A' , an *anti-homomorphism* of A into A' is a map $\sigma : A \rightarrow A'$ satisfying the following conditions:

1. $\sigma(a + b) = \sigma(a) + \sigma(b)$ for all $a, b \in A$;
2. $\sigma(1_A) = 1_{A'}$;
3. $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in A$.

If $A = A'$ and $\sigma : A \rightarrow A$ is a bijective anti-homomorphism, then σ is called an anti-automorphism. An *involution* is an anti-automorphism satisfying the additional condition:

4. $\sigma^2(a) = a$ for all $a \in A$.

If A is a central R -algebra, then an involution σ of A is said to be *of the first kind* if the restriction of σ to R is the identity map. We note that an involution of the first kind is an R -linear involution; that is, the map $\sigma : A \rightarrow A$ is an R -linear map.

For a central R -algebra A admitting an involution of the first kind, Braun [10, Thm. 5] gives a further characterisation of when A is an Azumaya algebra, which is stated in Theorem 1.5.7. The theorem shows that, in this setting, the condition that A is a finitely generated projective R -module is not required.

Theorem 1.5.7. *Let A be a central R -algebra admitting an involution of the first kind. Then A is an Azumaya algebra if and only if $\psi_A : A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A)$ is an R -linear isomorphism.*

Proof. See Braun [10, Thm. 5]. □

This theorem has been further generalised by Rowen [60, Cor. 1.7]. For a central R -algebra A admitting an involution of the first kind, the result of Rowen proves that for A to be an Azumaya algebra, it is sufficient to assume that ψ_A is an epimorphism.

1.6 The development of the theory of Azumaya algebras

The above results show some of the most general reformulations of the definition of an Azumaya algebra to date. We will now consider the historical development of these concepts, beginning with the 1951 paper of Azumaya [3].

In [3], Azumaya introduced the term “proper maximally central algebra” (p. 128). An R -algebra A which is free and finitely generated as a module over R is defined to be *proper maximally central* over R if $A \otimes A^{\text{op}}$ coincides with $\text{End}_R(A)$. It is known that a free R -module is both faithful and projective as a module over R , so this definition implies that A is faithfully projective as an R -module, and $A \otimes A^{\text{op}} \cong$

$\text{End}_R(A)$. This shows that the definition of a proper maximally central algebra is equivalent to Definition 1.4.1 under the additional assumption that A is free.

Assuming that R is a Noetherian ring, Auslander and Goldman prove Theorem 1.2.3 (see [2], Cor. 4.5 and Thm. 4.7). Further, Endo and Watanabe [21, Prop. 1.1] generalise this result by removing the Noetherian condition on R , as stated above in Theorem 1.2.3.

The equivalence of statements (1) and (2) of Theorem 1.5.3 was proven by Auslander and Goldman [2, Thm. 2.1] under the assumption that A is a central algebra over R , which we can see in Theorem 1.5.3 is not required. In Knus [41, Thm. III.5.1.1(2)], Theorem 1.5.3(2) has the condition that A is finitely generated, which isn't required for the equivalence of the statements. However, we note that given the equivalence of (1) and (2), the fact that A is finitely generated follows from Proposition 1.2.5.

Part (3) of Theorem 1.5.3 generalises a result of DeMeyer, Ingraham [17, Thm. II.3.4(2)]. Their result says that A is central over R and A is a progenerator over A^e if and only if A is an Azumaya algebra over R . But we can see that $\text{Hom}_{A^e}(A, -)$ being an exact functor is superfluous.

Azumaya [3, Thm. 15] proves the equivalence of Theorem 1.5.3 parts (1) and (5) with the extra condition that A is free over R . Further, with the assumption that A is projective, Bass [4, Thm. III.4.1] proves this equivalence. Theorem 1.5.3 shows that neither of these extra conditions on A are required for the equivalence of these two statements. Bass [4, Thm. III.4.1] shows that the definition of an Azumaya algebra is equivalent to the following:

There exists an R -algebra S and a faithfully projective R -module P such that $A \otimes_R S \cong \text{End}_R(P)$.

Comparing this with Theorem 1.5.3(6), the result of Knus [41, Thm. III.5.1.1(5)] refines this result by giving that S is a faithfully flat étale R -algebra.

We prove our main theorem of Chapter 3, Theorem 3.1.5, for Azumaya algebras which are free over their centres. So for an Azumaya algebra A as originally defined by Azumaya (with A free over its centre R), our Theorem 3.1.5 covers its K -theory.

Chapter 2

Algebraic K -Theory

Algebraic K -theory defines a sequence of functors $K_i(R)$, for $i \geq 0$, from the category of rings to the category of abelian groups. The lower K -groups K_0 , K_1 and K_2 were developed in the 1950s and 60s by Grothendieck, Bass and Milnor respectively. After much uncertainty, the “correct” definition of the higher K -groups was given by Quillen in 1974. For an introduction to the lower K -groups, see Magurn [48] or Silvester [64], and for an introduction to higher algebraic K -theory, see Rosenberg [58] or Weibel [70].

We begin this chapter by recalling the definitions of K_0 , K_1 and K_2 , and by observing some properties and examples of these lower K -groups. We then specialise to the lower K -groups of central simple algebras. Lastly, we look at the higher K -groups, and note some of their properties.

2.1 Lower K -groups

K_0

Let R be a ring. Let $\mathcal{P}\text{roj}(R)$ denote the monoid of isomorphism classes of finitely generated projective R -modules, with direct sum as the binary operation and the zero module as the identity element. Then $K_0(R)$ is defined to be the free abelian

group based on $\text{Proj}(R)$ modulo the subgroup generated by elements of the form $[P] + [Q] - [P \oplus Q]$, for $P, Q \in \text{Pr}(R)$. Alternatively, the group $K_0(R)$ can be defined as the group completion of the monoid $\text{Proj}(R)$, which is shown in [58, Thm. 1.1.3] to be equivalent to the definition given here. By [58, p. 5, Remarks], the group completion construction forms a functor from the category of abelian semigroups to the category of abelian groups.

Theorem 2.1.1. *For rings R and S , let T be an additive functor from the category of R -modules to the category of S -modules, with $T(R) \in \text{Pr}(S)$. Then T restricts to an exact functor from $\text{Pr}(R)$ to $\text{Pr}(S)$, which induces a group homomorphism $f : K_0(R) \rightarrow K_0(S)$ with $f([P]) = [T(P)]$ for each $P \in \text{Pr}(R)$.*

Proof. (See [48, Prop. 6.3].) If $P \in \text{Pr}(R)$, then $P \oplus Q \cong R^n$ for some R -module Q , and we have $T(P) \oplus T(Q) \cong T(R)^n$. Since $T(R)^n \in \text{Pr}(S)$, it follows that the functor T takes $\text{Pr}(R)$ to $\text{Pr}(S)$. All short exact sequences in $\text{Pr}(R)$ are split, so the restriction of T to $\text{Pr}(R) \rightarrow \text{Pr}(S)$ is an exact functor. Since functors preserve isomorphisms, T induces a monoid homomorphism

$$\text{Proj}(R) \longrightarrow \text{Proj}(S); [P] \longmapsto [T(P)].$$

Since the group completion is functorial, there is a group homomorphism $K_0(R) \rightarrow K_0(S); [P] \mapsto [T(P)]$. \square

For rings R and S , if $\phi : R \rightarrow S$ is a ring homomorphism, then S is a right R -module via ϕ . There is an additive functor

$$S \otimes_R - : R\text{-Mod} \longrightarrow S\text{-Mod}$$

which, by Theorem 2.1.1, induces a group homomorphism $f : K_0(R) \rightarrow K_0(S); f([P]) = [S \otimes_R P]$ for each $P \in \text{Pr}(R)$.

Theorem 2.1.2. *K_0 is a functor from the category of rings to the category of abelian groups.*

Proof. See [48, Thm. 6.22] for the details. \square

Theorem 2.1.3. *For rings R and S , there is a group isomorphism $K_0(R \times S) \cong K_0(R) \times K_0(S)$.*

Proof. See [48, Thm. 6.6]. \square

Recall that rings R and S are Morita equivalent, if $R\text{-Mod}$ and $S\text{-Mod}$ are equivalent as categories; that is, there are functors

$$R\text{-Mod} \begin{matrix} \xrightarrow{T} \\ \xleftarrow{U} \end{matrix} S\text{-Mod}$$

with natural equivalences $T \circ U \cong \text{id}_S$ and $U \circ T \cong \text{id}_R$. It follows from [49, Prop. II.10.2] that T and U are additive functors. In particular, $M_n(R)$ is Morita equivalent to R since the functors

$$\begin{aligned} R^n \otimes_{M_n(R)} - : M_n(R)\text{-Mod} &\longrightarrow R\text{-Mod} \\ \text{and} \quad R^n \otimes_R - : R\text{-Mod} &\longrightarrow M_n(R)\text{-Mod} \end{aligned}$$

form mutually inverse equivalences of categories.

Theorem 2.1.4. *For rings R and S , if R and S are Morita equivalent then $K_0(R) \cong K_0(S)$. In particular, $K_0(R) \cong K_0(M_n(R))$.*

Proof. (See [70, Cor. II.2.7.1].) This follows from Theorem 2.1.1 since the functors

$$R\text{-Mod} \begin{matrix} \xrightarrow{T} \\ \xleftarrow{U} \end{matrix} S\text{-Mod}$$

induce an equivalence between the categories $\mathcal{P}r(R)$ and $\mathcal{P}r(S)$, which induces a group isomorphism on the level of K_0 . \square

Examples 2.1.5. 1. If R is a field or a division ring, then $K_0(R) \cong \mathbb{Z}$. A finitely generated projective module over R is free, with a uniquely defined dimension.

Since any two finite dimensional free modules with the same dimension are isomorphic, we have $\mathcal{P}\text{roj}(R) \cong \mathbb{N}$, so $K_0(R) \cong \mathbb{Z}$.

2. Recall that a ring R is a semi-simple ring if it is semi-simple as a left module over itself; that is, R is a direct sum of simple R -submodules. By the Artin-Wedderburn theorem, R is isomorphic to $M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$, where each D_i is a division ring and each n_i is a positive integer. Then using Theorems 2.1.3 and 2.1.4,

$$K_0(R) \cong \bigoplus K_0(M_{n_i}(D_i)) \cong \bigoplus K_0(D_i) \cong \mathbb{Z}^r.$$

3. If R is a principal ideal domain (a commutative integral domain in which every ideal can be generated by a single element), or if R is a local ring, then $K_0(R) \cong \mathbb{Z}$. For a proof of these, see [58], Thm. 1.3.1 and Thm. 1.3.11 respectively.

K_1

Let $\text{GL}_n(R)$, the general linear group of R , denote the group of invertible $n \times n$ matrices with entries in R and matrix multiplication as the binary operation. Then $\text{GL}_n(R)$ is embedded into $\text{GL}_{n+1}(R)$ via

$$\begin{aligned} \text{GL}_n(R) &\longrightarrow \text{GL}_{n+1}(R) \\ M_n &\longmapsto \begin{pmatrix} M_n & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The union of the resulting sequence $\text{GL}_1(R) \subset \text{GL}_2(R) \subset \cdots \subset \text{GL}_n(R) \subset \cdots$ is called the infinite general linear group $\text{GL}(R) = \bigcup_{n=1}^{\infty} \text{GL}_n(R)$. Then $K_1(R)$ is defined to be the abelianisation of $\text{GL}(R)$; that is,

$$K_1(R) = \text{GL}(R)/[\text{GL}(R), \text{GL}(R)].$$

By [48, Prop. 9.3], the abelianisation construction forms a functor from the category of groups to the category of abelian groups.

The elementary matrix $e_{ij}(r)$, for $r \in R$ and $i \neq j$, $1 \leq i, j \leq n$, is defined to be the matrix in $\mathrm{GL}_n(R)$ which has 1's on the diagonal, r in the i - j entry and zeros elsewhere. Then $E_n(R)$ denotes the subgroup of $\mathrm{GL}_n(R)$ generated by all elementary matrices $e_{ij}(r)$ with $1 \leq i, j \leq n$. We note that the inverse of the matrix $e_{ij}(r)$ is $e_{ij}(-r)$ and the elementary matrices satisfy the following relations:

$$e_{ij}(r) e_{ij}(s) = e_{ij}(r + s)$$

$$[e_{ij}(r), e_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k, i \neq l \\ e_{il}(rs) & \text{if } j = k, i \neq l \\ e_{kj}(-sr) & \text{if } j \neq k, i = l. \end{cases}$$

Then $E_n(R)$ embeds in $E_{n+1}(R)$, and $E(R)$ is the infinite union of the $E_n(R)$. The Whitehead Lemma (see [48, Lemma 9.7]) states that $[\mathrm{GL}(R), \mathrm{GL}(R)] = E(R)$, so it follows that

$$K_1(R) = \mathrm{GL}(R)/E(R).$$

Theorem 2.1.6. *For rings R and S and a ring homomorphism $\phi : R \rightarrow S$, there is an induced group homomorphism $K_1(R) \rightarrow K_1(S)$ taking $(a_{ij})E(R)$ to $(\phi(a_{ij}))E(S)$. Then K_1 is a functor from the category of rings to the category of abelian groups.*

Proof. A ring homomorphism $\phi : R \rightarrow S$ defines a group homomorphism $\mathrm{GL}(R) \rightarrow \mathrm{GL}(S)$; $(a_{ij}) \mapsto (\phi(a_{ij}))$. Then there is an induced group homomorphism $K_1(R) \rightarrow K_1(S)$ taking $(a_{ij})E(R)$ to $(\phi(a_{ij}))E(S)$. See [48, Prop. 9.9] for the details. \square

Theorem 2.1.7. *For a ring R and a positive integer n , $K_1(M_n(R)) \cong K_1(R)$.*

Proof. (See the proof of [58, Thm. 1.2.4].) Since $M_k(M_n(R)) \cong M_{kn}(R)$, we have a

commutative diagram:

$$\begin{array}{ccc} \mathrm{GL}_k(M_n(R)) & \xrightarrow{\cong} & \mathrm{GL}_{kn}(R) \\ \downarrow & & \downarrow \\ \mathrm{GL}_{k+1}(M_n(R)) & \xrightarrow{\cong} & \mathrm{GL}_{(k+1)n}(R). \end{array}$$

It follows that $\mathrm{GL}(M_n(R)) \cong \mathrm{GL}(R)$, which induces the required isomorphism on the level of K_1 . \square

- Examples 2.1.8.** 1. If R is a field then $K_1(R) \cong R^*$. See [70, Eg. III.1.1.2] for the details. This follows since, for a commutative ring R , the determinant $\det : \mathrm{GL}(R) \rightarrow R^*$ induces a split surjective group homomorphism $\det : K_1(R) \rightarrow R^*$; $(a_{ij})E(R) \mapsto \det(a_{ij})$, with right inverse given by $R^* \rightarrow K_1(R)$; $a \mapsto aE(R)$. When R is a field, the kernel of the map \det is trivial, so $K_1(R) \cong R^*$.
2. If R is a division ring, then $K_1(R) \cong R^*/[R^*, R^*]$. See [70, Eg. III.1.3.5] for the details. This follows from the Dieudonné determinant (see [64, p. 124]). The Dieudonné determinant defines a surjective group homomorphism $\mathrm{Det} : \mathrm{GL}_n(R) \rightarrow R^*/[R^*, R^*]$ with kernel $E_n(R)$, such that

$$\mathrm{Det} \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} = \prod_i x_i [R^*, R^*]$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{\quad} & \mathrm{GL}_{n+1}(R) \\ & \searrow \mathrm{Det} & \swarrow \mathrm{Det} \\ & R^*/R' & \end{array}$$

It follows that for a division ring R , $K_1(R) \cong R^*/[R^*, R^*]$. If R is a field, then the Dieudonné determinant coincides with the usual determinant.

3. If R is a semi-local ring (subject to some conditions, detailed below), then $K_1(R) \cong R^*/[R^*, R^*]$. See [68, Thm. 2] for the details. Recall that R is a semi-local ring if $R/\text{rad}(R)$ is semi-simple, where $\text{rad}(R)$ denotes the Jacobson radical of R . By the Artin-Wedderburn Theorem, $R/\text{rad}(R)$ is isomorphic to a finite product of matrix rings over division rings. We assume that none of these matrix rings are isomorphic to $M_2(\mathbb{Z}/2\mathbb{Z})$ and that no more than one of the matrix rings has order 2. The required result follows since there is a Whitehead determinant which gives a surjective group homomorphism $R^* \rightarrow K_1(R)$. Vaserstein [68] shows that the kernel of this map is $[R^*, R^*]$, giving the required isomorphism.

K_2

For $n \geq 3$, the Steinberg group $\text{St}_n(R)$ of R is the group defined by generators $x_{ij}(r)$, with i, j a pair of distinct integers between 1 and n and $r \in R$, subject to the following relations which are called the Steinberg relations:

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$$

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k, i \neq l \\ x_{il}(rs) & \text{if } j = k, i \neq l \\ x_{kj}(-sr) & \text{if } j \neq k, i = l. \end{cases}$$

We observed on page 41 that the elementary matrices satisfy the Steinberg relations. So there is a surjective homomorphism $\text{St}_n(R) \rightarrow E_n(R)$ sending $x_{ij}(r)$ to $e_{ij}(r)$. As the Steinberg relations for $n+1$ include the Steinberg relations for n , there are natural maps $\text{St}_n(R) \rightarrow \text{St}_{n+1}(R)$. We write $\text{St}(R)$ for the direct limit $\varinjlim \text{St}_n(R)$. Then there is a canonical surjective map $\text{St}(R) \rightarrow E(R)$, and $K_2(R)$ is defined to be the kernel of this map.

Theorem 2.1.9. *K_2 is a functor from the category of rings to the category of abelian groups.*

Proof. See [48, Prop. 12.6]. □

Theorem 2.1.10. *For rings R and S , if R and S are Morita equivalent then $K_2(R) \cong K_2(S)$. In particular, since $M_n(R)$ is Morita equivalent to R , $K_2(R) \cong K_2(M_n(R))$.*

Proof. See [70, Cor. III.5.6.1]. □

Example 2.1.11. 1. If R is a finite field, then $K_2(R) = 0$. See [58, Cor. 4.3.13].

2. If R is a field, then Matsumoto's Theorem says that

$$K_2(R) = R^* \otimes_{\mathbb{Z}} R^* / \langle a \otimes (1 - a) : a \neq 0, 1 \rangle.$$

See [58, Thm. 4.3.15].

3. If R is a division ring, let U_R denote the group generated by $c(x, y)$, $x, y \in R^*$, subject to the relations:

$$(U0) \quad c(x, 1 - x) = 1 \quad (x \neq 1, 0),$$

$$(U1) \quad c(xy, z) = c(xy x^{-1}, x z x^{-1}) c(x, z)$$

$$(U2) \quad c(x, yz) c(y, zx) c(z, xy) = 1.$$

Then there is an exact sequence

$$0 \longrightarrow K_2(R) \longrightarrow U_R \longrightarrow [R^*, R^*] \longrightarrow 0$$

where $U_R \rightarrow [R^*, R^*]; c(x, y) \mapsto [x, y]$. See Rehmann [55, Cor. 2, p. 101].

2.2 Lower K -groups of central simple algebras

We now specialise to central simple algebras. In this section, all central simple algebras are assumed to be finite dimensional. Let F be a field and let A be a

central simple algebra over F . Clearly there is a ring homomorphism $F \rightarrow A$. Since K_0 is a functor from the category of rings to the category of abelian groups (see Theorem 2.1.2), we have an exact sequence

$$0 \longrightarrow \mathrm{ZK}_0(A) \longrightarrow K_0(F) \longrightarrow K_0(A) \longrightarrow \mathrm{CK}_0(A) \longrightarrow 0 \quad (2.1)$$

where $\mathrm{ZK}_0(A)$ and $\mathrm{CK}_0(A)$ are the kernel and cokernel of the map $K_0(F) \rightarrow K_0(A)$, respectively.

In this section, using the definition of K_0 , we will show what sequence (2.1) looks like. Using this, we will observe that $\mathrm{CK}_0(A)$ and $\mathrm{ZK}_0(A)$ are torsion abelian groups, and that CK_0 and ZK_0 are functors which do not respect Morita equivalence, from the category of central simple algebras over F to the category of abelian groups. We have a similar exact sequence for K_1 , and we show that the same results hold. In Chapter 3, we will generalise this result to cover Azumaya algebras which are free over their centres, and to cover all K_i groups for $i \geq 0$. This is the key to proving that $K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]$ for an Azumaya algebra A free over its centre R of dimension n (see Theorem 3.1.5).

K_0

Let A be a central simple algebra over a field F . Wedderburn's theorem says that A is isomorphic to $M_n(D)$ for a unique division algebra D and a unique positive integer n . Since K_0 is a functor from the category of rings to the category of abelian groups, we have $K_0(A) \cong K_0(M_n(D))$, and similarly for CK_0 and ZK_0 . Writing $M_n(D)$ instead of A , sequence (2.1) can be written as

$$0 \longrightarrow \mathrm{ZK}_0(M_n(D)) \longrightarrow K_0(F) \longrightarrow K_0(M_n(D)) \longrightarrow \mathrm{CK}_0(M_n(D)) \longrightarrow 0.$$

From the ring homomorphism $F \rightarrow M_n(D)$ and since $M_n(D)$ is Morita equivalent to D , there are induced functors

$$\begin{aligned} \mathcal{P}\mathbf{r}(F) &\longrightarrow \mathcal{P}\mathbf{r}(M_n(D)) \longrightarrow \mathcal{P}\mathbf{r}(D) \\ P \cong F^k &\longmapsto M_n(D)^k \longmapsto D^n \otimes_{M_n(D)} M_n(D)^k \cong D^{kn}, \end{aligned}$$

where every finitely generated projective module P over F is free. By Theorem 2.1.1, these induce group homomorphisms

$$\begin{aligned} K_0(F) &\xrightarrow{\gamma} K_0(M_n(D)) \xrightarrow{\delta} K_0(D) \\ [F^k] &\longmapsto [M_n(D)^k] \longmapsto [D^{kn}] \end{aligned}$$

where, for $[X] \in K_0(M_n(D))$, $\delta([X])$ is defined to be $[D^n \otimes_{M_n(D)} X]$.

Since F is a field and D is a division ring, from Example 2.1.5(1), $K_0(F) \cong \mathbb{Z}$; $[F^k] \mapsto k$ and $K_0(D) \cong \mathbb{Z}$; $[D^k] \mapsto k$. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ZK}_0(M_n(D)) & \longrightarrow & K_0(F) & \xrightarrow{\gamma} & K_0(M_n(D)) & \longrightarrow & \mathrm{CK}_0(M_n(D)) & \longrightarrow & 0 \\ & & \downarrow & & \mathrm{id} \downarrow & & \downarrow \delta & & \downarrow & & \\ 0 & \longrightarrow & \ker(\delta \circ \gamma) & \longrightarrow & K_0(F) & \xrightarrow{\delta \circ \gamma} & K_0(D) & \longrightarrow & \mathrm{coker}(\delta \circ \gamma) & \longrightarrow & 0 \\ & & \downarrow & & \cong \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\eta_n} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_n & \longrightarrow & 0 \end{array}$$

where the map $\eta_n : \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by $\eta_n(k) = kn$.

We know that $M_n(D)$ is Morita equivalent to D , so δ is an isomorphism and thus all the vertical maps in the above commutative diagram are isomorphisms. So the exact sequence (2.1) can be written as:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\eta_n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0 \quad (2.2)$$

since $\mathrm{ZK}_0(M_n(D)) \cong \ker(\eta_n) = 0$ and $\mathrm{CK}_0(M_n(D)) \cong \mathrm{coker}(\eta_n) = \mathbb{Z}_n$. A group G is said to be n -torsion if $x^n = e$ for all $x \in G$. We note that clearly both $\mathrm{ZK}_0(M_n(D))$ and $\mathrm{CK}_0(M_n(D))$ are n -torsion groups.

Let F be a fixed field. We will observe that CK_0 and ZK_0 form functors from the category of central simple algebras over F (with F -algebra homomorphisms) to the category of abelian groups. For any central simple algebra A over F , $\mathrm{CK}_0(A)$ is defined to be the cokernel of the map $K_0(F) \rightarrow K_0(A)$; $[P] \mapsto [A \otimes_F P]$, and ZK_0 is its kernel. They are clearly both abelian groups since $K_0(F)$ and $K_0(A)$ are abelian groups. For any two central simple algebras A and B over F with $\phi : A \rightarrow B$ an F -algebra homomorphism, then ϕ restricted to F gives the identity map on F . There is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ZK}_0(A) & \longrightarrow & K_0(F) & \longrightarrow & K_0(A) & \longrightarrow & \mathrm{CK}_0(A) & \longrightarrow & 0 \\ & & \downarrow & & \mathrm{id} \downarrow & & \downarrow \phi & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{ZK}_0(B) & \longrightarrow & K_0(F) & \longrightarrow & K_0(B) & \longrightarrow & \mathrm{CK}_0(B) & \longrightarrow & 0 \end{array}$$

where $\phi : K_0(A) \rightarrow K_0(B)$ is defined by $\phi([P]) = [B \otimes_A P]$ for each $[P] \in K_0(A)$. One can easily check that CK_0 and ZK_0 form the required functors.

We also observe that CK_0 does not respect Morita invariance. For a division algebra D over the field F , we know that D is Morita equivalent to $M_n(D)$. For D , the exact sequence (2.1) can be written as

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

since the homomorphism $K_0(F) \rightarrow K_0(D)$ maps $[F^k]$ to $[D \otimes_F F^k] = [D^k]$. From sequence (2.2) we saw that $\mathrm{CK}_0(M_n(D)) \cong \mathbb{Z}_n$, which is clearly not isomorphic to $\mathrm{CK}_0(D) = 0$.

K_1

Let A be a central simple algebra over a field F , such that A is isomorphic to $M_r(D)$ for a unique division algebra D and a unique positive integer r . Since K_1 is a functor from the category of rings to the category of abelian groups (see Theorem 2.1.6),

we will write $M_r(D)$ instead of A , since $K_1(A) \cong K_1(M_r(D))$. We have an exact sequence

$$1 \longrightarrow \mathrm{ZK}_1(M_r(D)) \longrightarrow K_1(F) \longrightarrow K_1(M_r(D)) \longrightarrow \mathrm{CK}_1(M_r(D)) \longrightarrow 1, \quad (2.3)$$

where $\mathrm{ZK}_1(M_r(D))$ and $\mathrm{CK}_1(M_r(D))$ are the kernel and cokernel of the map $K_1(F) \rightarrow K_1(M_r(D))$ respectively.

The ring homomorphism $F \rightarrow M_r(D); f \mapsto f \cdot I_r$ induces a map

$$\begin{aligned} \mathrm{GL}(F) &\longrightarrow \mathrm{GL}(M_r(D)) \\ (a_{ij}) &\longmapsto (a_{ij}I_r). \end{aligned}$$

Using Theorems 2.1.6 and 2.1.7, there are induced group homomorphisms

$$\begin{aligned} K_1(F) &\xrightarrow{\gamma} K_1(M_r(D)) && \xrightarrow{\delta} K_1(D) \\ (a_{ij})E(F) &\longmapsto (a_{ij}I_r)E(M_r(D)) \\ (d_{ij})E(M_r(D)) &\longmapsto (d_{ij})E(D). \end{aligned}$$

So we have a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{ZK}_1(M_r(D)) & \longrightarrow & K_1(F) & \xrightarrow{\gamma} & K_1(M_r(D)) & \longrightarrow & \mathrm{CK}_1(M_r(D)) & \longrightarrow & 1 \\ & & \downarrow & & \mathrm{id} \downarrow & & \downarrow \delta & & \downarrow & & \\ 1 & \longrightarrow & \ker(\delta \circ \gamma) & \longrightarrow & K_1(F) & \xrightarrow{\delta \circ \gamma} & K_1(D) & \longrightarrow & \mathrm{coker}(\delta \circ \gamma) & \longrightarrow & 1 \\ & & \downarrow & & \det \downarrow & & \downarrow \mathrm{Det} & & \downarrow & & \\ 1 & \longrightarrow & D' \cap F^{*r} & \longrightarrow & F^* & \xrightarrow{\beta} & D^*/D' & \longrightarrow & D^*/F^{*r}D' & \longrightarrow & 1. \end{array}$$

The maps \det and Det come from Examples 2.1.8 and the map $\beta : F^* \rightarrow D^*/D'$ is defined by $\beta(f) = \mathrm{Det} \circ (\delta \circ \gamma) \circ \det^{-1}(f) = f^r D'$, where $D' = [D^*, D^*]$.

We saw in Examples 2.1.8(1),(2) that $K_1(F) \cong F^*$ and $K_1(D) \cong D^*/D'$. So all the vertical maps in the above diagram are isomorphisms and the exact sequence

(2.3) can be written as

$$1 \longrightarrow D' \cap F^{*r} \longrightarrow F^* \longrightarrow D^*/D' \longrightarrow D^*/F^{*r}D' \longrightarrow 1. \quad (2.4)$$

We will show that the groups $\mathrm{ZK}_1(M_r(D)) \cong D' \cap F^{*r}$ and $\mathrm{CK}_1(M_r(D)) \cong D^*/F^{*r}D'$ are both nr -torsion groups for $n = \mathrm{ind}(D)$, where $\mathrm{ind}(D)$, the index of D over F , is defined to be the square root of the dimension of D over F . Note that by [63, Cor. 8.4.9], the dimension of D over F is a square number.

Lemma 2.2.1. *Let D be a division algebra with centre F of index n . Then for any $a \in D$, $a^n = \mathrm{Nrd}_D(a)d_a$ where $d_a \in D'$.*

Proof. Let $a \in D$ be arbitrary and let $f_a(x)$ be the minimal polynomial of a of degree m . By Wedderburn's Factorisation Theorem (see [42, Thm. 16.9]), $f_a(x) = (x - d_1ad_1^{-1}) \cdots (x - d_mad_m^{-1})$ where $d_i \in D$, and by [56, p. 124, Ex. 1], we have $f_a(x)^{n/m} = x^n - \mathrm{Trd}_D(a)x^{n-1} + \cdots + (-1)^n \mathrm{Nrd}_D(a)$. Then

$$\begin{aligned} \mathrm{Nrd}_D(a) &= (d_1ad_1^{-1} \cdots d_mad_m^{-1})^{(n/m)} \\ &= ([d_1, a]a[d_2, a]a \cdots a[d_m, a]a)^{(n/m)} \\ &= a^n d'_a \quad \text{where } d'_a \in D'. \end{aligned}$$

So $a^n = \mathrm{Nrd}_D(a)d_a$ for some $d_a \in D'$. □

For $a^r \in \mathrm{ZK}_1(M_r(D)) \cong D' \cap F^{*r}$, we have $a^r \in F^{*r} \subseteq F^*$, so $(a^r)^n = \mathrm{Nrd}_D(a^r)$ where $n = \mathrm{ind}(D)$. Since $a^r \in D'$, $\mathrm{Nrd}_D(a^r) = 1$, proving $\mathrm{ZK}_1(M_r(D))$ is nr -torsion. The equivalent result for $\mathrm{CK}_1(M_r(D))$ will follow from the previous lemma. For $\bar{a} \in \mathrm{CK}_1(M_r(D)) = D^*/F^{*r}D'$, we have $a \in D^*$ with $a^n = \mathrm{Nrd}_D(a)d_a$ for $d_a \in D'$ (by Lemma 2.2.1). Since $\mathrm{Nrd}_D : D^* \rightarrow F^*$, $\mathrm{Nrd}_D(a) \in F^*$ and therefore $a^n \in F^*D'$. Then $a^{nr} \in F^{*r}D'$, which shows $\bar{a}^{nr} = 1$, proving $\mathrm{CK}_1(M_r(D))$ is also nr -torsion.

Let F be a fixed field. As for K_0 , we will observe that CK_1 and ZK_1 form functors from the category of central simple algebras over F to the category of

abelian groups. For any two central simple algebras A and B over F with an F -algebra homomorphism $\phi : A \rightarrow B$, there is a commutative diagram:

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & \mathrm{ZK}_1(A) & \longrightarrow & K_1(F) & \longrightarrow & K_1(A) & \longrightarrow & \mathrm{CK}_1(A) & \longrightarrow & 1 \\
 & & \downarrow & & \mathrm{id} \downarrow & & \downarrow \phi & & \downarrow & & \\
 1 & \longrightarrow & \mathrm{ZK}_1(B) & \longrightarrow & K_1(F) & \longrightarrow & K_1(B) & \longrightarrow & \mathrm{CK}_1(B) & \longrightarrow & 1
 \end{array}$$

where $\phi : K_1(A) \rightarrow K_1(B)$ is defined by $\phi((a_{ij})E(A)) = (\phi(a_{ij}))E(B)$ for each $(a_{ij})E(A) \in K_1(A)$. It follows that CK_1 and ZK_1 are functors from the category of central simple algebras over F to the category of abelian groups.

We also observe that CK_1 does not respect Morita invariance. For a division algebra D over the field F , the exact sequence (2.3) can be written as

$$1 \longrightarrow D' \cap F^* \longrightarrow F^* \longrightarrow D^*/D' \longrightarrow D^*/F^*D' \longrightarrow 1$$

since the map $K_1(F) \rightarrow K_1(D)$ takes $(a_{ij})E(F)$ to $(a_{ij})E(D)$. From sequence (2.4) we saw that $\mathrm{CK}_1(M_r(D)) = D^*/F^{*r}D'$, but $\mathrm{CK}_1(D) = D^*/F^*D'$. In general they are not isomorphic (see for example [33, Eg. 7]).

Let us also mention another group which exhibits some similar properties to $\mathrm{CK}_1(D)$. For a central simple algebra A over a field F , the group $G(A) = A^*/(A^*)^2$, called the square class group of A , has been studied by Lewis and Tignol [45]. They note that the group $G(A)$ is a torsion abelian group of exponent two. They show that when A is a central simple algebra of odd degree over a field F , then the map $G(F) \rightarrow G(A)$ induced by inclusion is an isomorphism (see [45, Cor. 2, p. 367]). Although, in some aspects, the behaviour of $G(A)$ is similar to that of CK_1 , [45, Prop. 5] shows an aspect where they differ. It says that if D is a division ring and n is a positive integer greater than 2, then $G(M_n(D)) \cong G(D)$, which we observed above does not hold for $\mathrm{CK}_1(D)$.

2.3 Higher K -Theory

For higher algebraic K -theory, the K -groups were defined by Quillen in the early 1970s. Quillen gave two different constructions for higher K -theory, called the $+$ -construction and the Q -construction. The $+$ -construction defines the higher K -groups of a ring R . The Q -construction defines the K -groups of an exact category and, for a ring R , the K -groups $K_i(R)$ are defined to be the K -groups $K_i(\mathcal{P}r(R))$ where $\mathcal{P}r(R)$ is the category of finitely generated projective R -modules. The two constructions do in fact give the same K -groups for a ring R , although in appearance they are very different. (The proof is very involved, see [70, §IV.7].) For $i = 0, 1, 2$ the construction agrees with the definitions given in Section 2.1 (see [58, §5.2.1]). The K -groups, although complicated to define, are functorial in construction. We recall below some of their basic properties.

Example 2.3.1. (See [58, Thm. 5.3.2].) Let \mathbb{F}_q be a finite field with q elements. Then $K_0(\mathbb{F}_q) = \mathbb{Z}$ and for $i \geq 1$,

$$K_i(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}_{q^n-1} & \text{if } i = 2n - 1, \\ 0 & \text{if } i \text{ is even.} \end{cases}$$

Theorem 2.3.2. *If $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is an exact functor between exact categories, then \mathcal{F} induces a map $\mathcal{F}_* : K_i(\mathcal{C}) \rightarrow K_i(\mathcal{D})$. In particular, each K_i is a functor from the category of exact categories with exact functors to the category of abelian groups. Moreover, isomorphic functors induce the same map on the K -groups.*

Proof. See [70, p. IV.51], [54, p. 19]. □

Corollary 2.3.3. *For $i \geq 0$, each K_i is a functor from the category of rings to the category of abelian groups.*

Proof. See [70, §IV.1.1.2] or [58, Eg. 5.3.22]. This follows from Theorem 2.3.2, since a ring homomorphism $\phi : R \rightarrow S$ induces an exact functor $S \otimes_R - : \mathcal{P}r(R) \rightarrow \mathcal{P}r(S)$.

□

Theorem 2.3.4. *If the rings R and S are Morita equivalent, then $K_i(R) \cong K_i(S)$ for each $i \geq 0$.*

Proof. See [70, §IV.6.3.5]. This follows from Theorem 2.3.2, since if R and S are Morita equivalent then there is an equivalence of categories $\mathcal{P}r(R) \cong \mathcal{P}r(S)$. It follows that $K_i(R) \cong K_i(S)$ for each $i \geq 0$. \square

Let \mathcal{C} and \mathcal{D} be exact categories. The category of functors from \mathcal{C} to \mathcal{D} is an exact category which is denoted by $[\mathcal{C}, \mathcal{D}]$ and with morphisms defined to be natural transformations. Then by [54, p. 22], a sequence of functors from \mathcal{C} to \mathcal{D}

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of exact functors if for all $A \in \mathcal{C}$,

$$0 \longrightarrow \mathcal{F}'(A) \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}''(A) \longrightarrow 0$$

is an exact sequence in \mathcal{D} .

Theorem 2.3.5. *Let \mathcal{C} and \mathcal{D} be exact categories and let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of exact functors from \mathcal{C} to \mathcal{D} . Then

$$\mathcal{F}_* = \mathcal{F}'_* + \mathcal{F}''_* : K_i(\mathcal{C}) \longrightarrow K_i(\mathcal{D}).$$

Proof. See [54, Cor. 1, p. 22]. \square

In the above theorem, suppose $\mathcal{C} = \mathcal{D}$ is the category of finitely generated projective modules (or the category of graded finitely generated projective modules: see Section 4.5 for its definition). Take both \mathcal{F}' and \mathcal{F}'' to be the identity functor and

let $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}; A \mapsto A \oplus A$. Then clearly

$$0 \longrightarrow A \xrightarrow{\iota} A \oplus A \xrightarrow{\pi} A \longrightarrow 0$$

is an exact sequence, with maps $\iota : A \rightarrow A \oplus A; x \mapsto (x, 0)$ and $\pi : A \oplus A \rightarrow A; (x, y) \mapsto y$. So the homomorphism $\mathcal{F}_* : K_i(\mathcal{C}) \rightarrow K_i(\mathcal{C})$ induced by \mathcal{F} is

$$\mathcal{F}_* = \text{id} + \text{id} : K_i(\mathcal{C}) \longrightarrow K_i(\mathcal{C})$$

$$a \longmapsto a + a.$$

By induction, if $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}; A \mapsto A^k$ for $k \in \mathbb{N}$, then the induced homomorphism is $\mathcal{F}_* : K_i(\mathcal{C}) \rightarrow K_i(\mathcal{C}); a \mapsto ka$, which is also multiplication by k . We will use this result in the proofs of Propositions 3.1.3 and 5.4.3.

The following result was proven by Green et al. [24].

Theorem 2.3.6. *For a ring R , let $f : R \rightarrow R$ be an inner automorphism of R with $f(r) = a^{-1}ra$, where a is a unit of R . Then $K_i(f) : K_i(R) \rightarrow K_i(R)$ is the identity.*

Proof. (See [24, Lemma 2].) Since $f : R \rightarrow R$ is a ring homomorphism, there is a functor

$$F : \mathcal{P}\text{r}(R) \longrightarrow \mathcal{P}\text{r}(R); P \longmapsto R \otimes_f P,$$

where R is a right R -module via f and we note that $r \otimes r'p = rf(r') \otimes p$. We will show that there is a natural isomorphism ϕ from the identity functor to the functor F . For $P \in \mathcal{P}\text{r}(R)$, define

$$\phi_P : P \longrightarrow R \otimes_f P; p \longmapsto 1 \otimes ap.$$

Note that ϕ_P is an R -module homomorphism, since

$$\begin{aligned} \phi_P(rp) &= 1 \otimes arp = 1 \otimes ara^{-1}ap \\ &= 1 \otimes f^{-1}(r)ap = r \otimes ap = r(\phi_P(p)). \end{aligned}$$

Then for an R -module homomorphism $g : P \rightarrow P'$ in $\mathcal{Pr}(R)$, the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\phi_P} & R \otimes_f P \\ \text{id}(g) \downarrow & & \downarrow F(g)=1 \otimes g \\ P' & \xrightarrow{\phi_{P'}} & R \otimes_f P' \end{array}$$

So ϕ forms a natural transformation from the identity functor to F . Each ϕ_P is an isomorphism in $\mathcal{Pr}(R)$ since the map

$$\phi_P^{-1} : R \otimes_f P \longrightarrow P; \quad r \otimes p \longmapsto ra^{-1}p$$

is an R -module homomorphism which is an inverse of ϕ_P . So the inverses ϕ_P^{-1} form a natural transformation which is an inverse of ϕ . By applying Theorem 2.3.2, this shows that the induced map $K_i(R) \rightarrow K_i(R)$ is the identity. \square

The above theorem allowed Green et al. to prove the main theorem of their paper [24, Thm. 4], which shows that

$$K_i(D) \otimes \mathbb{Z}[1/n] \cong K_i(F) \otimes \mathbb{Z}[1/n]$$

for a division algebra D over F of dimension n^2 . In Chapter 3, we will generalise this result to cover Azumaya algebras which are free over their centres. Their proof uses that fact that $K_i(R) \rightarrow K_i(M_t(R)) \rightarrow K_i(R)$ is multiplication by t , where $R \rightarrow M_t(R)$ is the diagonal homomorphism $r \mapsto rI_t$ (see [24, Lemma 1]), and also the Skolem-Noether theorem which guarantees that algebra homomorphisms in the setting of central simple algebras are inner automorphisms. Their proof combines these results with Theorem 2.3.6 and with the main result of [16] which states that

$$\lim_{i \rightarrow \infty} M_{n^{2i}}(F) \cong \lim_{i \rightarrow \infty} M_{n^{2(i+1)}}(D).$$

Chapter 3

K-Theory of Azumaya Algebras

As we noted in the previous chapter, Green et al. [24] proved that for a division algebra finite dimensional over its centre, its K -theory is “essentially the same” as the K -theory of its centre; that is, for a division algebra D over its centre F of index n ,

$$K_i(D) \otimes \mathbb{Z}[1/n] \cong K_i(F) \otimes \mathbb{Z}[1/n]. \quad (3.1)$$

In this chapter, we prove that the isomorphism (3.1) holds for any Azumaya algebra free over its centre (see Theorem 3.1.5). A corollary of this is that the isomorphism holds for Azumaya algebras over semi-local rings. This extends the results of Hazrat (see [27, 28]) where (3.1) type properties have been proven for central simple algebras and Azumaya algebras over local rings respectively.

In Section 3.1, we introduce an abstract functor called a \mathcal{D} -functor, which is defined on the category of Azumaya algebras free over a fixed base ring. This is a continuation of [27] and [28] where Hazrat defines similar functors, over categories of central simple algebras and Azumaya algebras respectively. Here we show in Theorem 3.1.2 that the range of a \mathcal{D} -functor is the category of bounded torsion abelian groups. We then prove that the kernel and cokernel of the K -groups are \mathcal{D} -functors, which allows us to prove (3.1) type properties for Azumaya algebras which are free over their centres (see Theorem 3.1.5 and [31, Thm. 6]).

The Hochschild homology of Azumaya algebras behaves in a similar way to the

K -theory of Azumaya algebras. Cortiñas and Weibel [13] have shown that there is a similar result to Theorem 3.1.5 for the Hochschild homology of an Azumaya algebra. In Section 3.2, we begin by recalling the definition of Hochschild homology, which can be found in [47, 71]. We then recall the result of Cortiñas and Weibel which shows that $HH_*^k(A) \cong HH_*^k(R)$ for an R -Azumaya algebra A of constant rank.

3.1 \mathcal{D} -functors

Throughout this section, let R be a fixed commutative ring and $\mathcal{A}b$ be the category of abelian groups. Let $\text{Az}(R)$ be the category of Azumaya algebras free over R with R -algebra homomorphisms.

Definition 3.1.1. Consider a functor $\mathcal{F} : \text{Az}(R) \rightarrow \mathcal{A}b$; $A \mapsto \mathcal{F}(A)$. Such a functor is called a \mathcal{D} -functor if it satisfies the following three properties:

- (1) $\mathcal{F}(R)$ is the trivial group,
- (2) For any Azumaya algebra A free over R and any $k \in \mathbb{N}$, there is a homomorphism

$$\rho : \mathcal{F}(M_k(A)) \longrightarrow \mathcal{F}(A)$$

such that the composition $\mathcal{F}(A) \rightarrow \mathcal{F}(M_k(A)) \rightarrow \mathcal{F}(A)$ is η_k , where $\eta_k(x) = x^k$.

- (3) With ρ as in property (2), then $\ker(\rho)$ is k -torsion.

The name \mathcal{D} -functor comes from Dieudonné, since Dieudonné determinant behaves in a similar way to Property (2). Note that $M_k(A)$ is an Azumaya algebra free over R by Proposition 1.4.6, since A and $M_k(R)$ are Azumaya algebras free over R (see Example 1.4.5) and $M_k(A) \cong A \otimes_R M_k(R)$. Also note that the natural R -algebra homomorphism $A \rightarrow M_k(A)$; $a \mapsto aI_k$ induces a group homomorphism $\mathcal{F}(A) \rightarrow \mathcal{F}(M_k(A))$.

The following theorem shows that the range of a \mathcal{D} -functor is a bounded torsion abelian group.

Theorem 3.1.2. *Let A be an Azumaya algebra free over R of dimension n and let \mathcal{F} be a \mathcal{D} -functor. Then $\mathcal{F}(A)$ is n^2 -torsion.*

Proof. We begin by applying the definition of a \mathcal{D} -functor to the R -Azumaya algebra R . Property (1) says that $\mathcal{F}(R)$ is the trivial group and property (2) says that the composition $\mathcal{F}(R) \rightarrow \mathcal{F}(M_n(R)) \rightarrow \mathcal{F}(R)$ is η_n . So by the third property, $\mathcal{F}(M_n(R))$ is n -torsion. Since A is an Azumaya algebra free over R , we have $A \otimes_R A^{\text{op}} \cong \text{End}_R(A) \cong M_n(R)$, which means that $\mathcal{F}(A \otimes_R A^{\text{op}}) \cong \mathcal{F}(M_n(R))$ is n -torsion.

The natural R -algebra homomorphisms $A \rightarrow A \otimes_R A^{\text{op}}$, $A^{\text{op}} \rightarrow \text{End}_R(A^{\text{op}})$, $\text{End}_R(A^{\text{op}}) \cong M_n(R)$ and $A \otimes_R M_n(R) \cong M_n(A)$ combine to give the following R -algebra homomorphisms

$$A \longrightarrow A \otimes_R A^{\text{op}} \longrightarrow A \otimes \text{End}_R(A^{\text{op}}) \longrightarrow A \otimes M_n(R) \longrightarrow M_n(A).$$

These induce the homomorphisms $\mathcal{F}(A) \xrightarrow{i} \mathcal{F}(A \otimes_R A^{\text{op}}) \xrightarrow{r} \mathcal{F}(M_n(A))$. By property (2) in the definition of a \mathcal{D} -functor, we have a homomorphism $\rho : \mathcal{F}(M_n(A)) \rightarrow \mathcal{F}(A)$ such that the composition $\mathcal{F}(A) \rightarrow \mathcal{F}(M_n(A)) \rightarrow \mathcal{F}(A)$ is η_n . Consider the following diagram

$$\begin{array}{ccc} \mathcal{F}(A) & & \\ \downarrow i & \searrow \eta_n & \\ \mathcal{F}(A \otimes_R A^{\text{op}}) & & \\ \downarrow r & \xrightarrow{\rho} & \mathcal{F}(A) \end{array} \quad (3.2)$$

which is commutative by property (2). Let $a \in \mathcal{F}(A)$. Then

$$a^{n^2} = (\eta_n(a))^n = \rho \circ r((i(a))^n) = \rho \circ r(1) = 1$$

since $\mathcal{F}(A \otimes A^{\text{op}})$ is n -torsion. Thus $\mathcal{F}(A)$ is n^2 -torsion. \square

For a ring A with centre R , consider the inclusion $R \rightarrow A$. By Corollary 2.3.3, this induces the map $K_i(R) \rightarrow K_i(A)$ for $i \geq 0$. Consider the exact sequence

$$1 \longrightarrow \mathrm{ZK}_i(A) \longrightarrow K_i(R) \longrightarrow K_i(A) \longrightarrow \mathrm{CK}_i(A) \longrightarrow 1 \quad (3.3)$$

where ZK_i and CK_i are the kernel and cokernel of the map $K_i(R) \rightarrow K_i(A)$ respectively.

We will observe that CK_i can be considered as the following functor

$$\begin{aligned} \mathrm{CK}_i : \mathrm{Az}(R) &\longrightarrow \mathcal{A}b \\ A &\longmapsto \mathrm{CK}_i(A). \end{aligned}$$

For an Azumaya algebra A free over R , clearly $\mathrm{CK}_i(A) = \mathrm{coker}(K_i(R) \rightarrow K_i(A))$ is an abelian group. For a homomorphism $f : A \rightarrow A'$ of R -Azumaya algebras, there is an induced group homomorphism $f_* : K_i(A) \rightarrow K_i(A')$. The map f restricted to R is the identity map, so it induces the identity on the level of the K -groups. Since K_i preserves compositions of maps, the following diagram is commutative

$$\begin{array}{ccccc} K_i(R) & \longrightarrow & K_i(A) & \longrightarrow & \mathrm{CK}_i(A) \\ \mathrm{id} \downarrow & & \downarrow f_* & & \downarrow \mathrm{CK}_i(f) \\ K_i(R) & \longrightarrow & K_i(A') & \longrightarrow & \mathrm{CK}_i(A') \end{array}$$

Then it can be easily checked that CK_i forms the required functor. Similarly, we can consider ZK_i as the functor

$$\begin{aligned} \mathrm{ZK}_i : \mathrm{Az}(R) &\longrightarrow \mathcal{A}b \\ A &\longmapsto \mathrm{ZK}_i(A). \end{aligned}$$

Proposition 3.1.3. *With CK_i defined as above, CK_i is a \mathcal{D} -functor.*

Proof. Property (1) in the definition of a \mathcal{D} -functor is clear, since $K_i(Z(R)) \rightarrow K_i(R)$ is the identity map, and so clearly $\mathrm{CK}_i(R)$ is trivial.

To prove the second property, let A be an Azumaya algebra free over R and let $k \in \mathbb{N}$. Consider the functors

$$\begin{aligned} \mathcal{P}\mathcal{R}(A) &\xrightarrow{\phi} \mathcal{P}\mathcal{R}(M_k(A)) & \text{and} & & \mathcal{P}\mathcal{R}(M_k(A)) &\xrightarrow{\psi} \mathcal{P}\mathcal{R}(A) \\ X &\longmapsto M_k(A) \otimes_A X & & & Y &\longmapsto A^k \otimes_{M_k(A)} Y \end{aligned}$$

By Theorem 2.3.2, these functors induce homomorphisms ϕ and ψ on the level of the K -groups. Morita theory shows that ψ is an equivalence of categories (see page 39), so using Theorem 2.3.4, it induces an isomorphism from $K_i(M_k(A))$ to $K_i(A)$. For each $X \in \mathcal{P}\mathcal{R}(A)$, we have $\psi \circ \phi(X) \cong X^k$. Since K_i are functors which respect direct sums (see the remarks after Theorem 2.3.5), this composition induces a multiplication by k on the level of the K -groups. We have the following commutative diagram

$$\begin{array}{ccccccc} K_i(R) & \xrightarrow{f} & K_i(A) & \xrightarrow{\bar{f}} & \text{CK}_i(A) & \longrightarrow & 1 \\ \downarrow \text{id} & & \downarrow \phi & & \downarrow & & \\ K_i(R) & \xrightarrow{g} & K_i(M_k(A)) & \xrightarrow{\bar{g}} & \text{CK}_i(M_k(A)) & \longrightarrow & 1 \\ \downarrow \eta_k & & \cong \downarrow \psi & & \downarrow \rho & & \\ K_i(R) & \xrightarrow{f} & K_i(A) & \xrightarrow{\bar{f}} & \text{CK}_i(A) & \longrightarrow & 1 \end{array} \quad (3.4)$$

where compositions of columns are η_k , and thus property (2) holds.

Now let $x \in \text{CK}_i(M_k(A))$ such that $\rho(x) = 1$. Then there is $a \in K_i(M_k(A))$ with $\bar{g}(a) = x$ so that $1 = \bar{f} \circ \psi(a)$. As the rows are exact, there is $b \in K_i(R)$ with $f(b) = \psi(a)$. Taking powers of k , we have $f(b^k) = \psi(a^k)$ and $\bar{g}(a^k) = x^k$. As ψ is an isomorphism and $f \circ \eta_k(b) = \psi \circ g(b)$, it follows that $a^k = g(b)$. Then by the exactness of the rows, $\bar{g} \circ g(b) = 1$; that is, $x^k = 1$. So CK_i satisfies property (3) of a \mathcal{D} -functor. \square

Proposition 3.1.4. *With ZK_i defined as above, ZK_i is a \mathcal{D} -functor.*

Proof. This follows in exactly the same way as Proposition 3.1.3. \square

We are now in a position to prove that the K -theory of Azumaya algebras free

over their centres and the K -theory of Azumaya algebras over semi-local rings are isomorphic to the K -theory of their centres up to (their ranks) torsions.

Theorem 3.1.5. *Let A be an Azumaya algebra free over its centre R of dimension n . Then for any $i \geq 0$,*

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. Propositions 3.1.3 and 3.1.4 show that CK_i and ZK_i are both \mathcal{D} -functors. By Theorem 3.1.2, it follows that $CK_i(A)$ and $ZK_i(A)$ are n^2 -torsion abelian groups. Tensoring the exact sequence (3.3) by $\mathbb{Z}[1/n]$, since $CK_i(A) \otimes \mathbb{Z}[1/n]$ and $ZK_i(A) \otimes \mathbb{Z}[1/n]$ vanish, the result follows. \square

We recall the definition of a projective module of constant rank, which is used in the corollary below.

Definition 3.1.6. Let R be a commutative ring and let P be a prime ideal of R . Set $S = R \setminus P$ and write R_P for the localisation $S^{-1}R$, which is a local ring by [48, Ex. 6.45(i)]. (Recall a ring R is *local* if the non-invertible elements of R constitute a proper 2-sided ideal of R .) For an R -module M , we write $M_P = S^{-1}M$. If M is a finitely generated projective R -module, then, by [48, Prop. 6.44], M_P is a finitely generated projective module over R_P . Every finitely generated projective module over a local ring is free with a uniquely defined rank (see [58, Thm. 1.3.11]). Then $\text{rank}_P(M)$ is defined to be the rank of M_P as an R_P -module. We say that M is of *constant rank* if $\text{rank}_P(M) = n$ is the same for all prime ideals P and we write $\text{rank}(M) = n$. If M is a free module over the ring R , then M is of constant rank since $M_P \cong R_P \otimes_R M$ (see [48, Prop. 6.55]) and $\dim_R(M) = \dim_{R_P}(R_P \otimes_R M) = \text{rank}(M)$.

Corollary 3.1.7. *Let R be a semi-local ring and let A be an Azumaya algebra over its centre R of rank n . Then for any $i \geq 0$,*

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. Since A is finitely generated projective of constant rank and R is a semi-local ring, it follows that A is a free module over R (see [8], §II.5.3, Prop. 5), and thus the corollary follows from Theorem 3.1.5. \square

Theorem 3.1.5 covers Azumaya algebras which are free over their centres. We also mention here a theorem of Hazrat, Hoobler [29, Thm. 12], which shows that a similar result holds for the K -theory of Azumaya algebras over sheaves. Their result covers the case of Azumaya algebras over Noetherian centres, but it remains as a question whether this result holds for any Azumaya algebra of constant rank.

Question 3.1.8. Let A be an Azumaya algebra over its centre R of constant rank n . Then is it true that for any $i \geq 0$, $K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]$?

3.2 Homology of Azumaya algebras

We begin this section by recalling the definition of Hochschild homology, which can be found in [47, 71].

Let R be a ring. Recall that a chain complex of R -modules (C_*, d_*) is a family of right R -modules $\{C_n\}_{n \in \mathbb{Z}}$ together with R -module homomorphisms $d_n : C_n \rightarrow C_{n-1}$, such that the composition of any two consecutive maps is zero: $d_n \circ d_{n+1} = 0$ for all n . The maps d_n are called the differentials of C_* . The chain complex is usually written as:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \longrightarrow \cdots$$

The kernel of d_n is denoted by Z_n , and the image of d_{n+1} is denoted by B_n . Then for all n , $0 \subseteq B_n \subseteq Z_n \subseteq C_n$. The n^{th} homology module of C_* is the quotient

$$H_n(C_*) = Z_n/B_n.$$

Let K be a commutative ring, R be a K -algebra, and M an R -bimodule. We will write $R^{\otimes n}$ for the n -fold tensor product of R over K . The chain complex which

gives rise to Hochschild homology is given by $C_* = M \otimes R^{\otimes*}$ and is written as

$$\cdots \longrightarrow M \otimes R^{\otimes n} \xrightarrow{d_n} M \otimes R^{\otimes n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} M \otimes R \xrightarrow{d_1} M \longrightarrow 0.$$

Its differentials are $d_n = \sum_{i=0}^n (-1)^i \partial_i$ where ∂_i are defined by

$$\partial_i(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes a_2 \otimes \cdots \otimes a_n & \text{if } i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } 1 < i < n \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n \end{cases}$$

for $a_i \in R$ and $m \in M$. Then by [47, Lemma 1.1.2] $d_n \circ d_{n+1} = 0$, so (C_*, d_*) is a chain complex called the Hochschild complex. The n^{th} Hochschild homology module of R with coefficients in M is defined to be $H_n(M \otimes R^{\otimes*})$ and is denoted by $H_n(R, M)$. The direct sum $\bigoplus_{n \geq 0} H_n(R, M)$ is denoted by $H_*(R, M)$, and we will write $HH_*(R)$ for $H_*(R, R)$. By [47, §1.1.5], if R is commutative then $HH_*(R)$ is an R -module.

For a commutative ring R and an R -algebra A , suppose that K is a commutative subring of R . Then we consider the Hochschild homology, denoted by HH_*^K , of R and A as K -algebras.

The following result of Cortiñas and Weibel [13] shows that for an R -Azumaya algebra A of constant rank, its Hochschild homology behaves in a similar way to its K -theory (see Theorem 3.1.5).

Theorem 3.2.1. *Let A be an Azumaya algebra over a K -algebra R . If A has constant rank, then there is an isomorphism $HH_*^K(A) \cong HH_*^K(R)$.*

Proof. See [13, p. 53]. □

Chapter 4

Graded Azumaya Algebras

This chapter contains some basic definitions in the graded setting. These definitions can be found in [14, 39, 51], though not always in the generality that we require. We also include in this chapter a number of results in the graded setting which will be used in Chapter 5. In Section 4.1 we define graded rings and give the graded versions of the definitions of ideals, factor rings and ring homomorphisms. In Section 4.2 we give the corresponding definitions for graded modules. We then define graded division rings and show that a graded module over a graded division ring is graded free with a uniquely defined dimension.

In Section 4.3, we study graded central simple algebras graded by an arbitrary abelian group. We observe that the tensor product of two graded central simple R -algebras is graded central simple (Propositions 4.3.1 and 4.3.2). This result has been proven by Wall for $\mathbb{Z}/2\mathbb{Z}$ -graded central simple algebras (see [72, Thm. 2]), and by Hwang and Wadsworth for R -algebras with a totally ordered abelian, and hence torsion-free, grade group (see [39, Prop. 1.1]). We then observe that a graded central simple algebra, graded by an abelian group, is an Azumaya algebra (Theorem 4.3.3). This result extends the result of Boulagouaz [6, Prop. 5.1] and Hwang, Wadsworth [39, Cor. 1.2] (for a totally ordered abelian grade group) to graded rings in which the grade group is not totally ordered.

We define grading on matrices in Section 4.4, and observe some properties of

these graded matrix rings. We generalise a result of Caenepeel et al. [11, Thm. 2.1] in Theorem 4.4.9. We have also rewritten a number of known results on simple rings in the graded setting, which were required for the proof of this theorem. In Section 4.5, we define graded projective modules and graded Azumaya algebras. We prove that for a graded ring R , the graded matrix ring over R is Morita equivalent to R (see Proposition 4.5.4).

4.1 Graded rings

A ring $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ is called a Γ -graded ring, or simply a *graded ring*, if Γ is a group, each R_γ is an additive subgroup of R and $R_\gamma \cdot R_\delta \subseteq R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. We remark that initially Γ is an arbitrary group which is not necessarily abelian, so we will write Γ as a multiplicative group with identity element e . Each $x \in R$ can be uniquely expressed as a finite sum $x = \sum_{\gamma \in \Gamma} x_\gamma$ with each $x_\gamma \in R_\gamma$. For each $\gamma \in \Gamma$, the elements of R_γ are said to be *homogeneous of degree γ* and we write $\deg(r) = \gamma$ if $r \in R_\gamma$. We let $R^h = \bigcup_{\gamma \in \Gamma} R_\gamma$ be the set of homogeneous elements of R . The set

$$\Gamma_R = \{\gamma \in \Gamma : R_\gamma \neq \{0\}\},$$

which is also denoted by $\text{Supp}(R)$, is called the *support* (or *grade set*) of R . We note that the support of R is not necessarily a group.

Examples 4.1.1. 1. Let (Γ, \cdot) be a group and R be a ring. Set $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ where $R_e = R$ and for all $\gamma \in \Gamma$ with $\gamma \neq e$, let $R_\gamma = 0$. Then R can be considered as a trivially Γ -graded ring, with $\text{Supp}(R) = \{e\}$.

2. Let A be a ring. Then the polynomial ring $R = A[x]$ is a \mathbb{Z} -graded ring, with $R = \bigoplus_{n \in \mathbb{Z}} R_n$ where $R_n = Ax^n$ for $n \geq 0$ and $R_n = 0$ for $n < 0$. Here $\text{Supp}(R) = \mathbb{N} \cup \{0\}$.

3. Let (G, \cdot) be a group and let A be a ring. Then the group ring $R = A[G]$ is graded ring, with $R = \bigoplus_{g \in G} R_g$ where $R_g = \{ag : a \in A\}$, and $\text{Supp}(R) = G$.

Proposition 4.1.2. *Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ -graded ring. Then:*

1. 1_R is homogeneous of degree e ,
2. R_e is a subring of R ,
3. Each R_δ is an R_e -bimodule,
4. For an invertible element $r \in R_\delta$, its inverse r^{-1} is homogeneous of degree δ^{-1} .

Proof. (1) Suppose $1_R = \sum_{\gamma \in \Gamma} r_\gamma$ for $r_\gamma \in R_\gamma$. For some $\delta \in \Gamma$, let $s \in R_\delta$ be an arbitrary non-zero element. Then $s = s1_R = \sum_{\gamma \in \Gamma} sr_\gamma$ where $sr_\gamma \in R_{\delta\gamma}$ for all $\gamma \in \Gamma$. The decomposition is unique, so $sr_\gamma = 0$ for all $\gamma \in \Gamma$ with $\gamma \neq e$. But as s was arbitrary, this holds for all $s \in R$ (not necessarily homogeneous), and in particular $1_R r_\gamma = r_\gamma = 0$ if $\gamma \neq e$. For $\gamma = e$, we have $1_R = r_e$, so $1_R \in R_e$.

(2) This follows since R_e is an additive subgroup of R with $R_e R_e \subseteq R_e$ and $1 \in R_e$.

(3) This is immediate.

(4) Let $x = \sum_{\gamma} x_\gamma$ (with $\deg(x_\gamma) = \gamma$) be the inverse of r , so that $1 = rx = \sum_{\gamma} rx_\gamma$ where $rx_\gamma \in R_{\delta\gamma}$. Since 1 is homogeneous of degree e and the decomposition is unique, it follows that $rx_\gamma = 0$ for all $\gamma \neq \delta^{-1}$. Since r is invertible, we have $x_{\delta^{-1}} \neq 0$, so $x = x_{\delta^{-1}}$ as required. \square

We say that a Γ -graded ring $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ is a *strongly graded ring* if $R_\gamma R_\delta = R_{\gamma\delta}$ for all $\gamma, \delta \in \Gamma$. A graded ring R is called a *crossed product* if there is an invertible element in every homogeneous component R_γ of R ; that is, $R^* \cap R_\gamma \neq \emptyset$ for all $\gamma \in \Gamma$. For a Γ -graded ring R , let R^{h*} be the set of invertible homogeneous elements of R . Then R^{h*} is a subgroup of R^* , and the degree map $\deg : R^{h*} \rightarrow \Gamma$ is a group homomorphism. Let

$$\Gamma_R^* = \{\gamma \in \Gamma : R^* \cap R_\gamma \neq \emptyset\},$$

be the support of the invertible homogeneous elements of R . We note that R is a crossed product if and only if $\Gamma_R^* = \Gamma$, which is equivalent to the degree map being surjective.

Proposition 4.1.3. *Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ -graded ring. Then:*

1. *R is strongly graded if and only if $1 \in R_\gamma R_{\gamma^{-1}}$ for all $\gamma \in \Gamma$,*
2. *R is a crossed product if and only if the degree map is surjective,*
3. *If R is a crossed product, then R is a strongly graded ring.*

Proof. (1) The forward direction is immediate. Suppose $1 \in R_\gamma R_{\gamma^{-1}}$ for all $\gamma \in \Gamma$. Then for $\sigma, \delta \in \Gamma$,

$$R_{\sigma\delta} = R_e R_{\sigma\delta} = (R_\sigma R_{\sigma^{-1}}) R_{\sigma\delta} = R_\sigma (R_{\sigma^{-1}} R_{\sigma\delta}) \subseteq R_\sigma R_\delta$$

proving $R_{\sigma\delta} = R_\sigma R_\delta$, so R is strongly graded.

(2) This is immediate.

(3) For $\delta \in \Gamma$, there exists $r \in R^* \cap R_\delta$. So $r^{-1} \in R_{\delta^{-1}}$ and $1 = rr^{-1} \in R_\delta R_{\delta^{-1}}$.

□

A two-sided ideal I of R is called a *homogeneous ideal* (or *graded ideal*) if

$$I = \bigoplus_{\gamma \in \Gamma} (I \cap R_\gamma).$$

There are similar notions of graded subring, graded left ideal and graded right ideal. When I is a homogeneous ideal of R , the factor ring R/I forms a graded ring, with

$$R/I = \bigoplus_{\gamma \in \Gamma} (R/I)_\gamma \quad \text{where} \quad (R/I)_\gamma = (R_\gamma + I)/I.$$

A graded ring R is said to be *graded simple* if the only homogeneous two-sided ideals of R are $\{0\}$ and R .

Proposition 4.1.4. *An ideal I of a graded ring R is a homogeneous ideal if and only if I is generated as a two-sided ideal of R by homogeneous elements.*

Proof. Suppose $I = \bigoplus_{\gamma \in \Gamma} (I \cap R_\gamma)$, and consider $I \cap R^h$. This is a subset of I consisting of homogeneous elements. Any $x \in I$ can be written as $x = \sum_{\gamma \in \Gamma} x_\gamma$ where $x_\gamma \in I \cap R_\gamma$. So $x_\gamma \in I \cap R^h$, and thus x can be written as a sum of elements of $I \cap R^h$.

Conversely, suppose I is generated by the set $\{x_j\}_{j \in \mathcal{J}} \subseteq I$ consisting of homogeneous elements, for some indexing set \mathcal{J} . Then for any $x \in I$, we can write x as $x = \sum r_k x_j s_l$ where $r_k, s_l \in R^h$. Then for some r_k, x_j, s_l with $\alpha = \deg(r_k x_j s_l)$, we have $r_k x_j s_l \in I \cap R_\alpha$, since $x_j \in I$. This shows that $I = \bigoplus_{\gamma \in \Gamma} (I \cap R_\gamma)$, as required. \square

Let R and S be Γ -graded rings. Then a *graded ring homomorphism* $f : R \rightarrow S$ is a ring homomorphism such that $f(R_\gamma) \subseteq S_\gamma$ for all $\gamma \in \Gamma$. Further, f is called a *graded isomorphism* if f is bijective and, when such a graded isomorphism exists, we write $R \cong_{\text{gr}} S$. We remark that if f is a graded ring homomorphism which is bijective, then its inverse f^{-1} is also a graded ring homomorphism.

For a graded ring homomorphism $f : R \rightarrow S$, we know from the non-graded setting [37, p. 122] that $\ker(f)$ is an ideal of R and $\text{im}(f)$ is a subring of S . It can easily be shown that $\ker(f)$ is a graded ideal of R and $\text{im}(f)$ is a graded subring of S . Note that if Γ is an abelian group, then the centre $Z(R)$ of a graded ring R is a graded subring of R . If Γ is not abelian, then the centre of R may not be a graded subring of R , as is shown by the following example.

Example 4.1.5. Let $G = S_3 = \{e, a, b, c, d, f\}$ be the symmetric group of order 3, where

$$a = (23), \quad b = (13), \quad c = (12), \quad d = (123), \quad f = (132).$$

Let A be a ring, and consider the group ring $R = A[G]$, which is a G -graded ring by Example 4.1.1(3). Let $x = 1d + 1f \in R$, where $1 = 1_A$, and we note that x is not homogeneous in R . Then $x \in Z(R)$, but the homogeneous components of x are not in the centre of R . As x is expressed uniquely as the sum of homogeneous components, we have $x \notin \bigoplus_{g \in G} (Z(R) \cap R_g)$.

This example can be generalised by taking a non-abelian finite group G with a subgroup N which is normal and non-central. Let A be a ring and consider the group ring $R = A[G]$ as above. Then $x = \sum_{n \in N} 1n$ is in the centre of R , but the homogeneous components of x are not all in the centre of R .

4.2 Graded modules

Let Γ be a multiplicative group and let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a Γ -graded ring. We say that the group (Γ, \cdot) *acts freely* (as a left action) on a set Γ' if for all $\gamma, \gamma' \in \Gamma, \delta \in \Gamma'$, we have $\gamma\delta = \gamma'\delta$ implies $\gamma = \gamma'$, where $\gamma\delta$ denotes the image of δ under the action of γ . A Γ' -graded left R -module M is defined to be a left R -module with a direct sum decomposition $M = \bigoplus_{\gamma \in \Gamma'} M_\gamma$, where each M_γ is an additive subgroup of M and Γ acts freely on the set Γ' , such that $R_\gamma \cdot M_\lambda \subseteq M_{\gamma\lambda}$ for all $\gamma \in \Gamma, \lambda \in \Gamma'$. *From now on, unless otherwise stated, a graded module will mean a graded left module.*

We remark that here we define a graded R -module M which is graded by a set Γ' , where the grade group of R acts freely on Γ' . It is also possible to fix a group Γ and, for all graded rings and modules, to work instead with this fixed group Γ as the grade group. This is the approach taken throughout the book [51]. While our approach is more general than taking a fixed group Γ , in most cases their weaker approach suffices.

As for graded rings, $M^h = \bigcup_{\gamma \in \Gamma'} M_\gamma$ and $\Gamma'_M = \{\gamma \in \Gamma' : M_\gamma \neq \{0\}\}$. A *graded submodule* N of M is defined to be an R -submodule such that $N = \bigoplus_{\gamma \in \Gamma'} (N \cap M_\gamma)$. For a graded submodule N of M , we define the graded quotient structure on M/N by

$$M/N = \bigoplus_{\gamma \in \Gamma'} (M/N)_\gamma \quad \text{where} \quad (M/N)_\gamma = (M_\gamma + N)/N.$$

A graded R -module M is said to be *graded simple* if the only graded submodules of M are $\{0\}$ and M . A *graded free R -module* M is defined to be a graded R -module which is free as an R -module with a homogeneous base.

Let $N = \bigoplus_{\gamma \in \Gamma''} N_\gamma$ be another graded R -module, such that there is a set Δ

containing Γ' and Γ'' as subsets, where Γ acts freely on Δ . The graded R -module M can be written as $M = \bigoplus_{\gamma \in \Delta} M_\gamma$ with $M_\gamma = 0$ if $\gamma \in \Delta \setminus \Gamma'_M$, and similarly for N . A *graded R -module homomorphism* $f : M \rightarrow N$ is an R -module homomorphism such that $f(M_\delta) \subseteq N_\delta$ for all $\delta \in \Delta$. Let $\text{Hom}_{R\text{-gr-Mod}}(M, N)$ denote the group of graded R -module homomorphisms, which is an additive subgroup of $\text{Hom}_R(M, N)$. If the graded R -module homomorphism f is bijective, then f is called a *graded isomorphism* and we write $M \cong_{\text{gr}} N$. If f is a bijective graded R -module homomorphism, then its inverse f^{-1} is also a graded R -module homomorphism.

Suppose Δ is a group containing Γ' and Γ'' as subgroups, where the group Γ acts freely on Δ , and suppose R , M and N are defined as above. A graded R -module homomorphism from M to N may also shift the grading on N . For each $\delta \in \Delta$, we have a subgroup of $\text{Hom}_R(M, N)$ of δ -shifted homomorphisms

$$\text{Hom}_R(M, N)_\delta = \{f \in \text{Hom}_R(M, N) : f(M_\gamma) \subseteq N_{\gamma\delta} \text{ for all } \gamma \in \Delta\}.$$

For some $\delta \in \Delta$, we define the δ -shifted R -module $M(\delta)$ as $M(\delta) = \bigoplus_{\gamma \in \Delta} M(\delta)_\gamma$ where $M(\delta)_\gamma = M_{\gamma\delta}$. Then

$$\text{Hom}_R(M, N)_\delta = \text{Hom}_{R\text{-gr-Mod}}(M, N(\delta)) = \text{Hom}_{R\text{-gr-Mod}}(M(\delta^{-1}), N).$$

Let $\text{HOM}_R(M, N) = \bigoplus_{\delta \in \Delta} \text{Hom}_R(M, N)_\delta$.

Let M, N be Δ -graded R -modules as above. Then $M \oplus N$ forms a Δ -graded R -module with

$$M \oplus N = \bigoplus_{\gamma \in \Delta} (M \oplus N)_\gamma \quad \text{where } (M \oplus N)_\gamma = M_\gamma \oplus N_\gamma.$$

With R defined as above and for $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$, consider

$$R^n(d) = R(\delta_1) \oplus \dots \oplus R(\delta_n) = \bigoplus_{\gamma \in \Gamma} (R(\delta_1)_\gamma \oplus \dots \oplus R(\delta_n)_\gamma)$$

where $R(\delta_i)_\gamma = R_{\gamma\delta_i}$ is the γ -component of the δ_i -shifted graded R -module $R(\delta_i)$. Note that for each i , with $1 \leq i \leq n$, the element e_i of the standard basis for $R^n(d)$ is homogeneous of degree δ_i^{-1} . Similarly for $\{\delta_i\}_{i \in I}$ where I is an indexing set and $\delta_i \in \Gamma$, consider the Γ -graded R -module $\bigoplus_{i \in I} R(\delta_i)$. Again the i -th basis element e_i of $\bigoplus_{i \in I} R(\delta_i)$ is homogeneous of degree δ_i^{-1} .

Suppose F is a Γ -graded R -module which is graded free with a homogeneous base $\{b_i\}_{i \in I}$, where $\deg(b_i) = \delta_i$. Then the map

$$\begin{aligned} \varphi : \bigoplus_{i \in I} R(\delta_i^{-1}) &\longrightarrow F \\ e_i &\longmapsto b_i \end{aligned}$$

is a graded R -module isomorphism, and we write $F \cong_{\text{gr}} \bigoplus_{i \in I} R(\delta_i^{-1})$. Conversely, suppose F is a Γ -graded R -module with $\bigoplus_{i \in I} R(\delta_i^{-1}) \cong_{\text{gr}} F$, as graded R -modules, for some $\{\delta_i\}_{i \in I}$ with $\delta_i \in \Gamma$. Since $\{e_i\}_{i \in I}$ forms a homogeneous basis of $\bigoplus_{i \in I} R(\delta_i^{-1})$, the images of the e_i under the graded isomorphism form a homogeneous basis for F . Thus F is graded free if and only if $F \cong_{\text{gr}} \bigoplus_{i \in I} R(\delta_i^{-1})$ for some $\{\delta_i\}_{i \in I}$ with $\delta_i \in \Gamma$. In the same way, F is graded free with a finite basis if and only if $F \cong_{\text{gr}} R^n(d)$ for some $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$.

Proposition 4.2.1. *Let R be a Γ -graded ring and let $\delta \in \Gamma$. Then $R(\delta) \cong_{\text{gr}} R$ as graded R -modules if and only if $\delta \in \Gamma_R^*$.*

Proof. If $\delta \in \Gamma_R^*$, then let $x \in R^* \cap R_\delta$. Then there is a graded R -module isomorphism $\mathcal{R}_x : R \rightarrow R(\delta); r \mapsto rx$. Conversely, if $\phi : R \cong_{\text{gr}} R(\delta)$, then $\phi(1) \in R_\delta$ with inverse $\phi^{-1}(1)$, so $\phi(1) \in R^* \cap R_\delta$. \square

We note that it follows from the above proposition that $R(\delta) \cong_{\text{gr}} R$ for all $\delta \in \Gamma$ if and only if R is a crossed product. For a graded ring R and $\delta, \alpha \in \Gamma$, it is clear that $R(\alpha) \cong_{\text{gr}} R(\delta)$ as graded R -modules if and only if $R(\alpha)(\delta^{-1}) \cong_{\text{gr}} R(\delta)(\delta^{-1})$; that is, if and only if $R(\delta^{-1}\alpha) \cong_{\text{gr}} R$. Then by Proposition 4.2.1, $R(\alpha) \cong_{\text{gr}} R(\delta)$ as graded R -modules if and only if $\delta^{-1}\alpha \in \Gamma_R^*$.

For a Γ -graded ring R , consider $M_{n \times m}(R)$, the set of $n \times m$ matrices over R . For $n = m$, in Section 4.4, we will define grading on the matrix ring $M_n(R)$, which gives $M_n(R)$ the structure of a graded ring. Here we define shifted matrices $M_{n \times m}(R)$, which will be used in the following proposition. Note that they do not have the structure of either graded rings or graded R -modules. For $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$, $(a) = (\alpha_1, \dots, \alpha_m) \in \Gamma^m$, let

$$M_{n \times m}(R)[d][a] = \begin{pmatrix} R_{\delta_1^{-1}\alpha_1} & R_{\delta_1^{-1}\alpha_2} & \cdots & R_{\delta_1^{-1}\alpha_m} \\ R_{\delta_2^{-1}\alpha_1} & R_{\delta_2^{-1}\alpha_2} & \cdots & R_{\delta_2^{-1}\alpha_m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\delta_n^{-1}\alpha_1} & R_{\delta_n^{-1}\alpha_2} & \cdots & R_{\delta_n^{-1}\alpha_m} \end{pmatrix}$$

So $M_{n \times m}(R)[d][a]$ consists of matrices with the ij -entry in $R_{\delta_i^{-1}\alpha_j}$. Further, we let $\text{GL}_{n \times m}(R)[d][a]$ denote the set of invertible $n \times m$ matrices with shifting as above. The following proposition extends Proposition 4.2.1. A similar argument will be used in the proof of Proposition 5.4.3.

Proposition 4.2.2. *Let R be a Γ -graded ring and let $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$, $(a) = (\alpha_1, \dots, \alpha_m) \in \Gamma^m$. Then $R^n(d) \cong_{\text{gr}} R^m(a)$ as graded R -modules if and only if there exists $(r_{ij}) \in \text{GL}_{n \times m}(R)[d][a]$.*

Proof. If $r = (r_{ij}) \in \text{GL}_{n \times m}(R)[d][a]$, then there is a graded R -module homomorphism

$$\begin{aligned} \mathcal{R}_r : R^n(d) &\longrightarrow R^m(a) \\ (x_1, \dots, x_n) &\longmapsto (x_1, \dots, x_n)r. \end{aligned}$$

Since r is invertible, there is a matrix $t \in \text{GL}_{m \times n}(R)$ with $rt = I_n$ and $tr = I_m$. So there is an R -module homomorphism $\mathcal{R}_t : R^m(a) \rightarrow R^n(d)$, which is an inverse of \mathcal{R}_r . This proves that \mathcal{R}_r is bijective, and therefore it is a graded R -module isomorphism.

Conversely, if $\phi : R^n(d) \cong_{\text{gr}} R^m(a)$, then we can construct a matrix as follows. Let e_i denote the basis element of $R^n(d)$ with 1 in the i -th entry and zeros elsewhere.

Then let $\phi(e_i) = (r_{i1}, r_{i2}, \dots, r_{im})$, and let $r = (r_{ij})_{n \times m}$. It can be easily verified that $r \in M_{n \times m}(R)[d][a]$. In the same way, using $\phi^{-1} : R^m(a) \rightarrow R^n(d)$ construct a matrix t . Let e'_i denote the i -th element of the standard basis for $R^m(a)$. Since

$$e_i = \phi^{-1} \circ \phi(e_i) = r_{i1}\phi^{-1}(e'_1) + r_{i2}\phi^{-1}(e'_2) + \dots + r_{im}\phi^{-1}(e'_m)$$

for each i , and in a similar way for $\phi \circ \phi^{-1}$, we can show that $rt = I_n$ and $tr = I_m$. So $(r_{ij}) \in \text{GL}_{n \times m}(R)[d][a]$. \square

For convenience, in the above definition of $M_{n \times m}(R)[d][a]$, if $(a) = (e, \dots, e)$, we will write $M_{n \times m}(R)[d]$ instead of $M_{n \times m}(R)[d][e]$. We let

$$\Gamma_{M_{n \times m}(R)}^* = \{(d) \in \Gamma^n : \text{GL}_{n \times m}(R)[d] \neq \emptyset\}.$$

Then it is immediate from the above proposition that $R^n(d) \cong_{\text{gr}} R^n$ as graded R -modules if and only if $(d) \in \Gamma_{M_n(R)}^*$.

A Γ -graded ring $D = \bigoplus_{\gamma \in \Gamma} D_\gamma$ is called a *graded division ring* if every non-zero homogeneous element has a multiplicative inverse. It follows from Proposition 4.1.2(4) that Γ_D is a group, so we can write $D = \bigoplus_{\gamma \in \Gamma_D} D_\gamma$. Then, as a Γ_D -graded ring, D is a crossed product and it follows from Proposition 4.1.3(3) that D is strongly Γ_D -graded.

Examples 4.2.3. 1. Let E be a division ring and let $D = E[x, x^{-1}]$ be the Laurent polynomials over E . Then D is a graded division ring, with $D = \bigoplus_{n \in \mathbb{Z}} D_n$ where $D_n = \{ax^n : a \in E\}$.

2. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the real quaternion algebra, with multiplication defined by $i^2 = -1$, $j^2 = -1$ and $ij = -ji = k$. It is known that \mathbb{H} is a non-commutative division ring with centre \mathbb{R} . We note that \mathbb{H} can be given two different graded division ring structures, with grade groups \mathbb{Z}_2 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ respectively.

For \mathbb{Z}_2 : Let $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$. Then $\mathbb{H} = \mathbb{C}_0 \oplus \mathbb{C}_1$, where $\mathbb{C}_0 = \mathbb{C}$ and $\mathbb{C}_1 = \mathbb{C}j$.

For $\mathbb{Z}_2 \times \mathbb{Z}_2$: Let $\mathbb{H} = R_{(0,0)} \oplus R_{(1,0)} \oplus R_{(0,1)} \oplus R_{(1,1)}$, where

$$R_{(0,0)} = \mathbb{R}, \quad R_{(1,0)} = \mathbb{R}i, \quad R_{(0,1)} = \mathbb{R}j, \quad R_{(1,1)} = \mathbb{R}k.$$

In both cases, it is routine to show \mathbb{H} that forms a graded division ring.

In the following propositions we are considering graded modules over graded division rings. We note that the grade groups here are defined as above; that is, we do not initially assume them to be abelian or torsion-free. The proofs follow the standard proofs in the non-graded setting (see [37, §IV, Thms. 2.4, 2.7, 2.13]), or the graded setting (see [5, Thm. 3], [51, Prop. 4.6.1], [39, p. 79]); however extra care needs to be given since the grading is neither abelian nor torsion-free.

Proposition 4.2.4. *Let Γ be a group which acts freely on a set Γ' . Let R be a Γ -graded division ring and M be a Γ' -graded module over R . Then M is a graded free R -module. More generally, any linearly independent subset of M consisting of homogeneous elements can be extended to form a homogeneous basis of M .*

Proof. Note that the first statement is an immediate consequence of the second, since for any $m \in M^h$, $\{m\}$ is a linearly independent subset of M . Fix a linearly independent subset X of M consisting of homogeneous elements. Let

$$F = \{Q \subseteq M^h : X \subseteq Q \text{ and } Q \text{ is } R\text{-linearly independent}\}.$$

This is a non-empty partially ordered set with inclusion, and every chain $Q_1 \subseteq Q_2 \subseteq \dots$ in F has an upper bound $\bigcup Q_i \in F$. By Zorn's Lemma, F has a maximal element, which we denote by P . If $\langle P \rangle \neq M$, then there is a homogeneous element $m \in M^h \setminus \langle P \rangle$. We will show that $P \cup \{m\}$ is a linearly independent set containing X , contradicting our choice of P .

Suppose $rm + \sum r_i p_i = 0$, where $r, r_i \in R$, $p_i \in P$ and $r \neq 0$. Since $r \neq 0$ there is a homogeneous component of r , say r_λ , which is also non-zero. Considering the $\lambda \deg(m)$ -homogeneous component of this sum, we have $m = r_\lambda^{-1} r_\lambda m = -\sum r_\lambda^{-1} r_i p_i$

for r_i homogeneous, which contradicts our choice of m . Hence $r = 0$, which implies each $r_i = 0$. This gives the required contradiction, so $M = \langle P \rangle$, completing the proof. \square

Proposition 4.2.5. *Let Γ be a group which acts freely on a set Γ' . Let R be a Γ -graded division ring and M be a Γ' -graded module over R . Then any two homogeneous bases of M over R have the same cardinality.*

Proof. Suppose M has an infinite basis Z . Since R is in particular a ring, we can apply [37, Thm. IV.2.6], which shows that every basis of M has the same cardinality as Z . We now assume that M has two homogeneous bases X and Y , where X and Y are finite. Then $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$ for $x_i, y_i \in M^h \setminus 0$. As X is a basis for M , we can write

$$y_m = r_1 x_1 + \dots + r_n x_n$$

for some $r_i \in R^h$, where $\deg(y_m) = \deg(r_i) \deg(x_i)$ for each i . Since $y_m \neq 0$, we have at least one $r_i \neq 0$. Let r_k be the first non-zero r_i , and we note that r_k is invertible as it is non-zero and homogeneous in R . Then

$$x_k = r_k^{-1} y_m - r_k^{-1} r_{k+1} x_{k+1} - \dots - r_k^{-1} r_n x_n,$$

and the set $X' = \{y_m, x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n\}$ spans M since X spans M , where X' consists of homogeneous elements. So

$$y_{m-1} = s_m y_m + t_1 x_1 + \dots + t_{k-1} x_{k-1} + t_{k+1} x_{k+1} + \dots + t_n x_n$$

for $s_m, t_i \in R^h$. There is at least one non-zero t_i , since if all the t_i are zero, then either y_m and y_{m-1} are linearly dependent or y_{m-1} is zero. Let t_j denote the first non-zero t_i . Then x_j can be written as a linear combination of y_{m-1} , y_m and those x_i with $i \neq j, k$. Therefore the set $X'' = \{y_{m-1}, y_m\} \cup \{x_i : i \neq j, k\}$ spans M since X' spans M . We can write y_{m-2} as a linear combination of the elements of X'' .

Continuing this process of adding a y and removing an x gives, after the k -th step, a set which spans M consisting of $y_m, y_{m-1}, \dots, y_{m-k+1}$ and $n - k$ of the x_i . If $n < m$, then after the n -th step, we would have that the set $\{y_m, \dots, y_{m-n+1}\}$ spans M . But if $n < m$, then $m - n + 1 \geq 2$, so this set does not contain y_1 , and therefore y_1 can be written as a linear combination of the elements of this set. This contradicts the linear independence of Y , so we must have $m \leq n$. Repeating a similar argument with X and Y interchanged gives $n \leq m$, so $n = m$. \square

The propositions above say that for a graded module M over a graded division ring R , M has a homogeneous basis and any two homogeneous bases of M have the same cardinality. The cardinal number of any homogeneous basis of M is called the dimension of M over R , and it is denoted by $\dim_R(M)$.

Proposition 4.2.6. *Let Γ be a group which acts freely on a set Γ' . Let R be a Γ -graded division ring and M be a Γ' -graded module over R . If N is a graded submodule of M , then*

$$\dim_R(N) + \dim_R(M/N) = \dim_R(M).$$

Proof. Let Y be a homogeneous basis of N . Then Y is a linearly independent subset of M consisting of homogeneous elements, so using Proposition 4.2.4, there is a homogeneous basis X of M containing Y . We will show that $U = \{x + N : x \in X \setminus Y\}$ is a homogeneous basis of M/N . Note that clearly U consists of homogeneous elements. Let $m + N \in (M/N)^h$. Then $m \in M^h$ and $m = \sum r_i x_i + \sum s_j y_j$ where $r_i, s_j \in R$, $y_j \in Y$ and $x_i \in X \setminus Y$. So $m + N = \sum r_i(x_i + N)$, which shows that U spans M/N . If $\sum r_i(x_i + N) = 0$, for $r_i \in R$, $x_i \in X \setminus Y$, then $\sum r_i x_i \in N$ which implies that $\sum r_i x_i = \sum s_k y_k$ for $s_k \in R$ and $y_k \in Y$. But $X = Y \cup (X \setminus Y)$ is linearly independent, so $r_i = 0$ and $s_k = 0$ for all i, k . Therefore U is a homogeneous basis for M/N and as we can construct a bijective map $U \rightarrow X \setminus Y$, we have $|U| = |X \setminus Y|$. Then $\dim_R M = |X| = |Y| + |X \setminus Y| = |Y| + |U| = \dim_R N + \dim_R(M/N)$. \square

4.3 Graded central simple algebras

Let Γ be a multiplicative group and let R be a commutative Γ -graded ring. A Γ -graded R -algebra $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ is defined to be a graded ring which is an R -algebra such that the associated ring homomorphism φ is a graded homomorphism, where $\varphi : R \rightarrow A$ with $\varphi(R) \subseteq Z(A)$. Graded R -algebra homomorphisms and graded subalgebras are defined analogously to the equivalent terms for graded rings or graded modules.

Let A and B be graded R -algebras, such that $\Gamma_A \subseteq Z_\Gamma(\Gamma_B)$, where $Z_\Gamma(\Gamma_B)$ is the set of elements of Γ which commute with Γ_B . Then $A \otimes_R B$ has a natural grading as a graded R -algebra given by $A \otimes_R B = \bigoplus_{\gamma \in \Gamma} (A \otimes_R B)_\gamma$ where:

$$(A \otimes_R B)_\gamma = \left\{ \sum_i a_i \otimes b_i : a_i \in A^h, b_i \in B^h, \deg(a_i) \deg(b_i) = \gamma \right\}$$

Note that the condition $\Gamma_A \subseteq Z_\Gamma(\Gamma_B)$ is needed to ensure that the multiplication on $A \otimes_R B$ is well defined.

Let R be a Γ -graded ring, M be a Γ -graded R -module and let $\text{END}_R(M) = \text{HOM}_R(M, M)$, where $\text{HOM}_R(M, M)$ is defined on page 69. The ring $\text{END}_R(M)$ is a Γ -graded ring with the usual addition and with multiplication defined as $g \cdot f = f \circ g$ for all $f, g \in \text{END}_R(M)$. If M is graded free with a finite homogeneous base, then we have $\text{END}_R(M) = \text{End}_R(M)$ (see [51, Remark 2.10.6(ii)]). In fact, if M and N are Γ -graded R -modules with M finitely generated, then $\text{HOM}_R(M, N) = \text{Hom}_R(M, N)$ (see [51, Cor. 2.4.4]). Further, if R is a commutative Γ -graded ring and $\Gamma_R \subseteq Z_\Gamma(\Gamma_M)$, then $\text{End}_R(M)$ is a Γ -graded R -algebra.

A *graded field* is defined to be a commutative graded division ring. Note that the support of a graded field is an abelian group. Let R be a graded field. A graded algebra A over R is said to be a *graded central simple algebra* over R if A is graded simple as a graded ring, $Z(A) \cong_{\text{gr}} R$ and $[A : R] < \infty$. Note that since the centre of A is R , which is a graded field, A is graded free as a graded module over its centre by Proposition 4.2.4, so the dimension of A over R is uniquely defined.

For a Γ -graded ring A , let A^{op} denote the opposite graded ring, where the grade group of A^{op} is the opposite group Γ^{op} . So for a graded R -algebra A , in order to define $A \otimes_R A^{\text{op}}$, we note that the support of A must be abelian. Moreover for the following propositions, we require that the group Γ is an abelian group (see Theorem 4.3.5 and the preceding comments). *Since the grade groups are assumed to be abelian for the remainder of this section, we will write them as additive groups.*

By combining Propositions 4.3.1 and 4.3.2, we show that the tensor product of two graded central simple R -algebras is graded central simple, where the grade group Γ , as below, is abelian but not necessarily torsion-free. This has been proven by Wall for graded central simple algebras with $\mathbb{Z}/2\mathbb{Z}$ as the support (see [72, Thm. 2]), and by Hwang and Wadsworth for R -algebras with a torsion-free grade group (see [39, Prop. 1.1]).

Proposition 4.3.1. *Let Γ be an abelian group. Let R be a Γ -graded field and let A and B be Γ -graded R -algebras. If A is graded central simple over R and B is graded simple, then $A \otimes_R B$ is graded simple.*

Proof. Let I be a homogeneous two-sided ideal of $A \otimes B$, with $I \neq 0$. We will show that $A \otimes B = I$. Note that since I is a homogeneous ideal, by Proposition 4.1.4, it is generated by homogeneous elements. First suppose $a \otimes b$ is a homogeneous element of I , where $a \in A^h$ and $b \in B^h$. Then A is the homogeneous two-sided ideal generated by a , so there exist $a_i, a'_i \in A^h$ with $1 = \sum a_i a'_i$. Then

$$\sum (a_i \otimes 1)(a \otimes b)(a'_i \otimes 1) = 1 \otimes b$$

is an element of I . Similarly, B is the homogeneous two-sided ideal generated by b . Repeating the above argument shows that $1 \otimes 1$ is an element of I , proving $I = A \otimes B$ in this case.

Now suppose there is an element $x \in I^h$, where $x = a_1 \otimes b_1 + \cdots + a_k \otimes b_k$, with $a_j \in A^h$, $b_j \in B^h$ and k as small as possible. Note that since x is homogeneous, $\deg(a_j) + \deg(b_j) = \deg(x)$ for all j . By the above argument we can suppose that

$k > 1$. As above, since $a_k \in A^h$, there are $c_i, c'_i \in A^h$ with $1 = \sum c_i a_k c'_i$. Then

$$\sum (c_i \otimes 1)x(c'_i \otimes 1) = \left(\sum (c_i a_1 c'_i)\right) \otimes b_1 + \cdots + \left(\sum (c_i a_{k-1} c'_i)\right) \otimes b_{k-1} + 1 \otimes b_k,$$

where the terms $(\sum_i (c_i a_j c'_i)) \otimes b_j$ are homogeneous elements of $A \otimes B$. Thus, without loss of generality, we can assume that $a_k = 1$. Then a_k and a_{k-1} are linearly independent, since if $a_{k-1} = r a_k$ with $r \in R$, then $a_{k-1} \otimes b_{k-1} + a_k \otimes b_k = a_k \otimes (r b_{k-1} + b_k)$, which is homogeneous and thus gives a smaller value of k .

Thus $a_{k-1} \notin R = Z(A)$, and so there is a homogeneous element $a \in A$ with $aa_{k-1} - a_{k-1}a \neq 0$. Consider the commutator

$$(a \otimes 1)x - x(a \otimes 1) = (aa_1 - a_1a) \otimes b_1 + \cdots + (aa_{k-1} - a_{k-1}a) \otimes b_{k-1},$$

where the last summand is not zero. If the whole sum is not zero, then we have constructed a homogeneous element in I with a smaller k . Otherwise suppose the whole sum is zero, and write $c = aa_{k-1} - a_{k-1}a$. Then we can write $c \otimes b_{k-1} = \sum_{j=1}^{k-2} x_j \otimes b_j$ where $x_j = -(aa_j - a_ja)$. Since $0 \neq c \in A^h$ and A is the homogeneous two-sided ideal generated by c , using the same argument as above, we have

$$1 \otimes b_{k-1} = x'_1 \otimes b_1 + \cdots + x'_{k-2} \otimes b_{k-2} \tag{4.1}$$

for some $x'_j \in A^h$. Since b_1, \dots, b_{k-1} are linearly independent homogeneous elements of B , they can be extended to form a homogeneous basis of B , say $\{b_i\}$, by Proposition 4.2.4. Then $\{1 \otimes b_i\}$ forms a homogeneous basis of $A \otimes_R B$ as a graded A -module, so in particular they are A -linearly independent, which is a contradiction to equation (4.1). This reduces the proof to the first case. \square

Proposition 4.3.2. *Let Γ be an abelian group. Let R be a Γ -graded field and let A and B be Γ -graded R -algebras. If $A' \subseteq A$ and $B' \subseteq B$ are graded subalgebras, then*

$$Z_{A \otimes B}(A' \otimes B') = Z_A(A') \otimes Z_B(B').$$

In particular, if A and B are central over R , then $A \otimes_R B$ is central.

Proof. First note that by Proposition 4.2.4, $A', B', Z_A(A')$ and $Z_B(B')$ are graded free over R , so they are flat over R . Thus we can consider $Z_{A \otimes B}(A' \otimes B')$ and $Z_A(A') \otimes Z_B(B')$ as graded subalgebras of $A \otimes B$.

The inclusion \supseteq follows immediately. For the reverse inclusion, let $x \in Z_{A \otimes B}(A' \otimes B')$. Let $\{b_1, \dots, b_n\}$ be a homogeneous basis for B over R which exists thanks to Proposition 4.2.4. Then x can be written uniquely as $x = x_1 \otimes b_1 + \dots + x_n \otimes b_n$ for $x_i \in A$ (see [37, Thm. IV.5.11]). For every $a \in A'$, $(a \otimes 1)x = x(a \otimes 1)$, so

$$(ax_1) \otimes b_1 + \dots + (ax_n) \otimes b_n = (x_1 a) \otimes b_1 + \dots + (x_n a) \otimes b_n.$$

By the uniqueness of this representation we have $x_i a = ax_i$, so that $x_i \in Z_A(A')$ for each i . Thus we have shown that $x \in Z_A(A') \otimes_R B$. Similarly, let $\{c_1, \dots, c_k\}$ be a homogeneous basis of $Z_A(A')$. Then we can write x uniquely as $x = c_1 \otimes y_1 + \dots + c_k \otimes y_k$ for $y_i \in B$. A similar argument to above shows that $y_i \in Z_B(B')$, completing the proof. \square

In the following theorem, since Γ is an abelian group, we define multiplication in $\text{End}_R(A)$ to be $g \cdot f = g \circ f$.

Theorem 4.3.3. *Let Γ be an abelian group. Let A be a Γ -graded central simple algebra over a Γ -graded field R . Then A is an Azumaya algebra over R .*

Proof. Since A is graded free of finite dimension over R , it follows that A is faithfully projective over R . There is a natural graded R -algebra homomorphism $\psi : A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A)$ defined by $\psi(a \otimes b)(x) = axb$ where $a, x \in A$, $b \in A^{\text{op}}$. Since graded ideals of A^{op} are the same as graded ideals of A , we have that A^{op} is graded simple. By Proposition 4.3.1, $A \otimes A^{\text{op}}$ is also graded simple, so ψ is injective. Hence the map is surjective by dimension count, using Theorem 4.2.6. This shows that A is an Azumaya algebra over R , as required. \square

Corollary 4.3.4. *Let Γ be an abelian group. Let A be a Γ -graded central simple algebra over a Γ -graded field R of dimension n . Then for any $i \geq 0$,*

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. By Theorem 4.3.3, a graded central simple algebra A over R is an Azumaya algebra. From Proposition 4.2.4, since R is a graded field, A is a free R -module. The corollary now follows immediately from Theorem 3.1.5, since A is an Azumaya algebra free over its centre. \square

For a graded field R , Theorem 4.3.3 shows that a graded central simple R -algebra, graded by an abelian group Γ , is an Azumaya algebra over R . This theorem can not be extended to cover non-abelian grading. Consider a finite dimensional division algebra D and a group G . The group ring DG is a graded division ring, so is clearly a graded simple algebra, and if G is abelian, Theorem 4.3.3 implies that DG is an Azumaya algebra. However in general, for an arbitrary group G , DG is not always an Azumaya algebra. DeMeyer and Janusz [18] have proven the following theorem.

Theorem 4.3.5. *Let R be a ring and let G be a group. Then the group ring RG is an Azumaya algebra if and only if the following three conditions hold:*

1. *R is an Azumaya algebra,*
2. *$[G : Z(G)] < \infty$,*
3. *the commutator subgroup of G has finite order m and m is invertible in R .*

Proof. See [18, Thm. 1]. \square

4.4 Graded matrix rings

Let Γ be a group and R be a Γ -graded ring. We will write Γ as a multiplicative group, since Γ is not necessarily abelian. Throughout this section, unless otherwise

stated, we will assume that all graded rings, graded modules and graded algebras are also Γ -graded. Let $\lambda \in \Gamma$ and $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$. Let $M_n(R)(d)_\lambda$ denote the $n \times n$ -matrices over R with the degree shifted as follows:

$$M_n(R)(d)_\lambda = \begin{pmatrix} R_{\delta_1 \lambda \delta_1^{-1}} & R_{\delta_1 \lambda \delta_2^{-1}} & \cdots & R_{\delta_1 \lambda \delta_n^{-1}} \\ R_{\delta_2 \lambda \delta_1^{-1}} & R_{\delta_2 \lambda \delta_2^{-1}} & \cdots & R_{\delta_2 \lambda \delta_n^{-1}} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\delta_n \lambda \delta_1^{-1}} & R_{\delta_n \lambda \delta_2^{-1}} & \cdots & R_{\delta_n \lambda \delta_n^{-1}} \end{pmatrix}$$

Thus $M_n(R)(d)_\lambda$ consists of matrices with the ij -entry in $R_{\delta_i \lambda \delta_j^{-1}}$.

Proposition 4.4.1. *With the notation as above, there is a graded ring*

$$M_n(R)(d) = \bigoplus_{\lambda \in \Gamma} M_n(R)(d)_\lambda.$$

Proof. (See [51, Prop. 2.10.4].) For any $\lambda_1, \lambda_2 \in \Gamma$,

$$M_n(R)(d)_{\lambda_1} M_n(R)(d)_{\lambda_2} \subseteq M_n(R)(d)_{\lambda_1 \lambda_2}.$$

For any i, j with $1 \leq i, j \leq n$, $R_{\delta_i \lambda \delta_j^{-1}} \cap \sum_{\gamma \neq \lambda} R_{\delta_i \gamma \delta_j^{-1}} = 0$, so

$$M_n(R)(d)_\lambda \cap \left(\sum_{\gamma \neq \lambda} M_n(R)(d)_\gamma \right) = 0.$$

Any matrix in $M_n(R)$ can be written as a sum of matrices with a homogeneous element in one entry and zeros elsewhere. If $A \in M_n(R)$ has $a \in R_\epsilon$ in the ij -entry and zeros elsewhere, then taking $\lambda = \delta_i^{-1} \epsilon \delta_j$ gives that $A \in M_n(R)(d)_\lambda$, and hence any matrix in $M_n(R)$ can be written as an element of $M_n(R)(d)$. \square

Let M be a graded R -module which is graded free with a finite homogeneous base $\{b_1, \dots, b_n\}$, where $\deg(b_i) = \delta_i$. We noted in the previous section that the ring $\text{End}_R(M)$ is a graded ring with multiplication defined as $g \cdot f = f \circ g$ for all $f, g \in \text{End}_R(M)$. If we ignore the grading, it is well-known that $\text{End}_R(M) \cong M_n(R)$

as rings (see [37, Thm. VII.1.2]). When we take the grading into account, the following proposition shows that this isomorphism is in fact a graded isomorphism.

Proposition 4.4.2. *Let M be a graded free Γ -graded R -module with a finite homogeneous base $\{b_1, \dots, b_n\}$, where $\deg(b_i) = \delta_i$. Then $\text{End}_R(M) \cong_{\text{gr}} M_n(R)(d)$ as Γ -graded rings, where $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$.*

Proof. (See [51, Prop. 2.10.5].) The remarks before the proposition show that $\text{End}_R(M) \cong M_n(R)$. To show that the isomorphism is graded, let $f \in \text{End}_R(M)_\lambda$ for some $\lambda \in \Gamma$. Then as in the non-graded setting, there are elements $r_{ij} \in R$ such that

$$\begin{aligned} f(b_1) &= r_{11}b_1 + r_{12}b_2 + \cdots + r_{1n}b_n \\ f(b_2) &= r_{21}b_1 + r_{22}b_2 + \cdots + r_{2n}b_n \\ &\vdots \\ f(b_n) &= r_{n1}b_1 + r_{n2}b_2 + \cdots + r_{nn}b_n, \end{aligned}$$

with associated matrix $(r_{ij}) \in M_n(R)$. Since we have $f(b_i) \in M_{\delta_i\lambda}$ for each b_i , it follows that each r_{ij} is homogeneous of degree $\delta_i\lambda\delta_j^{-1}$. So $(r_{ij}) \in M_n(R)(d)_\lambda$ as required. \square

Remark 4.4.3. Note that above all graded R -modules are considered as left modules. For the λ -component of the graded matrix ring $M_n(R)(d)$, we set the degree of the ij -entry to be $\delta_i\lambda\delta_j^{-1}$. Then in the above proposition, we defined the multiplication in $\text{End}_R(M)$ as $g \cdot f = f \circ g$ to ensure that the isomorphism is a graded ring isomorphism.

If we define multiplication in $\text{End}_R(M)$ by $g \cdot f = g \circ f$, then in the non-graded setting, we have a ring isomorphism $\text{End}_R(M) \rightarrow M_n(R^{\text{op}})$, where this map is the composition of the homomorphism in the above proposition with the transpose map. To make $\text{End}_R(M)$, with this multiplication, into a graded ring, it will be graded by Γ^{op} . In the above proposition, we will have $\text{End}_R(M) \cong_{\text{gr}} M_n(R^{\text{op}})(d)$, where

$M_n(R^{\text{op}})(d)$ is also Γ^{op} -graded and the degree of the ij -entry of $M_n(R^{\text{op}})(d)$ is defined to be $\delta_i^{-1} \cdot_{\text{op}} \lambda \cdot_{\text{op}} \delta_j$.

In Definition 4.5.3 and in Section 5.4, since Γ is an abelian group and R is a commutative graded ring, we will define multiplication in endomorphism rings to be $g \cdot f = g \circ f$. Then we will use the grading on matrix rings mentioned in the previous paragraph.

Suppose M is a graded right R -module and multiplication in $\text{End}_R(M)$ is defined by $g \cdot f = g \circ f$ (see page 85 for some comments on graded right R -modules). Then to get a graded ring isomorphism $\text{End}_R(M) \cong_{\text{gr}} M_n(R)(d)$, we need to define the grading on the matrix ring $M_n(R)(d)$ as having its ij -entry in the λ -component of degree $\delta_i^{-1} \lambda \delta_j$.

With a graded R -module M defined as in the above proposition, we define the graded R -module homomorphisms E_{ij} by $E_{ij}(b_l) = \delta_{il} b_j$ for $1 \leq i, j, l \leq n$, where δ_{il} is the Kronecker delta. We note that the graded R -module homomorphisms E_{ij} , for $1 \leq i, j, l \leq n$, are homogeneous elements in $\text{End}_R(M)$ of degree $\delta_i^{-1} \delta_j$. Then $\{E_{ij} : 1 \leq i, j \leq n\}$ forms a homogeneous basis for $\text{End}_R(M)$. Via the isomorphism $\text{End}_R(M) \cong_{\text{gr}} M_n(R)(d)$, the map E_{ij} corresponds to the matrix e_{ij} , where e_{ij} is the matrix with 1 in the ij -entry and zeros elsewhere. This observation will be used in the following proposition to prove some basic properties of the graded matrix rings.

Proposition 4.4.4. *Let R be a Γ -graded ring and let $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$.*

1. *If $\pi \in S_n$ is a permutation, then*

$$M_n(R)(\delta_1, \dots, \delta_n) \cong_{\text{gr}} M_n(R)(\delta_{\pi(1)}, \dots, \delta_{\pi(n)}).$$

2. *If $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ with each $\gamma_i \in \Gamma_R^*$, then*

$$M_n(R)(\delta_1, \dots, \delta_n) \cong_{\text{gr}} M_n(R)(\gamma_1 \delta_1, \dots, \gamma_n \delta_n).$$

If R is a graded division ring, then any set of $(\gamma_1, \dots, \gamma_n) \in \Gamma_R^n$ can be chosen.

3. If $\sigma \in Z(\Gamma)$, the centre of Γ , then

$$M_n(R)(\delta_1, \dots, \delta_n) = M_n(R)(\delta_1\sigma, \dots, \delta_n\sigma).$$

Proof. (See [39, p. 78], [51, Remarks 2.10.6].) We observed on page 69 that

$$M = R(\delta_1^{-1}) \oplus \dots \oplus R(\delta_n^{-1})$$

has a homogeneous basis $\{e_1, \dots, e_n\}$ with $\deg(e_i) = \delta_i$. So from the above proposition, we have $\text{End}_R(M) \cong_{\text{gr}} M_n(R)(\delta_1, \dots, \delta_n)$, where the map E_{ij} corresponds to the matrix e_{ij} .

(1) Let $\delta_{\pi(i)} = \tau_i$. Then $N = R(\tau_1^{-1}) \oplus \dots \oplus R(\tau_n^{-1})$ has a standard homogeneous basis $\{e'_1, \dots, e'_n\}$ with $\deg(e'_i) = \tau_i$, and we have

$$\text{End}_R(N) \cong_{\text{gr}} M_n(R)(\tau_1, \dots, \tau_n).$$

We define a graded R -module isomorphism $\phi : M \rightarrow N$ by $\phi(e_i) = e'_{\pi^{-1}(i)}$, which induces a graded isomorphism $\phi : \text{End}_R(M) \rightarrow \text{End}_R(N)$; $f \mapsto \phi \circ f \circ \phi^{-1}$. Combining these graded isomorphisms gives the required result:

$$M_n(R)(\delta_1, \dots, \delta_n) \cong_{\text{gr}} \text{End}_A(M) \xrightarrow{\phi} \text{End}_R(N) \cong_{\text{gr}} M_n(R)(\delta_{\pi(1)}, \dots, \delta_{\pi(n)}).$$

(2) Let $u_1, \dots, u_n \in R$ be homogeneous units of R with $\deg(u_i) = \gamma_i$. Let $N = R(\delta_1^{-1}) \oplus \dots \oplus R(\delta_n^{-1})$, where we are considering N with the homogeneous basis $\{u_1 e_1, \dots, u_n e_n\}$ with $\deg(u_i e_i) = \gamma_i \delta_i$. Then with this basis,

$$\text{End}_R(N) \cong_{\text{gr}} M_n(R)(\gamma_1 \delta_1, \dots, \gamma_n \delta_n).$$

Define a graded R -module isomorphism $\phi : M \rightarrow N$ by $\phi(e_i) = u_i^{-1}(u_i e_i)$. The required isomorphism follows in a similar way to (1). Clearly, if R is a graded division ring then every non-zero homogeneous element is invertible, so any set of

$(\gamma_1, \dots, \gamma_n) \in \Gamma_R^n$ can be chosen here.

(3) Since $\sigma \in Z(\Gamma)$, it is clear that $M_n(R)(\delta_1, \dots, \delta_n)_\lambda = M_n(R)(\delta_1\sigma, \dots, \delta_n\sigma)_\lambda$ for each $\lambda \in \Gamma$, as required. \square

We note that if R is a Γ -graded ring, then for a Γ -graded right R -module M , we have $M_\lambda R_\gamma \subseteq M_{\lambda\gamma}$ for all $\lambda, \gamma \in \Gamma$. Then as for graded left R -module homomorphisms, graded right R -module homomorphisms between graded right R -modules may shift the grading. We note that they are left shifted; that is, if M, N are graded right R -modules and $f \in \text{Hom}_R(M, N)_\delta$, then $f(M_\gamma) \subseteq N_{\delta\gamma}$ for all $\gamma \in \Gamma$. The δ -shifted right R -module $(\delta)M$ is defined as $(\delta)M = \bigoplus_{\gamma \in \Gamma} ((\delta)M)_\gamma$ where $((\delta)M)_\gamma = M_{\delta\gamma}$.

The following four results (Proposition 4.4.5 to Corollary 4.4.8) are the graded versions of some results on simple rings (see [37, §IX.1]). These are required for the proof of Theorem 4.4.9.

Proposition 4.4.5. *Let D be a graded division ring, let V be a finite dimensional graded module over D , and let $R = \text{End}_D(V)$. If A and B are graded right R -modules which are faithful and graded simple, then $(\gamma)A \cong_{\text{gr}} B$ as graded right R -modules for some $\gamma \in \Gamma$.*

Proof. From Proposition 4.4.2, $R \cong_{\text{gr}} M_n(D)(d)$ for some $(d) \in \Gamma^n$. Then we will show that $M_n(D)(d)$ contains a minimal graded right ideal. For some i , consider $J_1 = \{e_{i,i}X : X \in M_n(D)(d)\}$, where $e_{i,i}$ is the elementary matrix with 1 in the i, i -entry. Then J_1 consists of all matrices in $M_n(D)(d)$ with j -th row zero for $j \neq i$ and J_1 is a graded right ideal of $M_n(D)(d)$. If J_1 is not minimal, there is a non-zero graded right ideal J_2 of $M_n(D)(d)$, with $J_2 \subseteq J_1$. Since J_2 is a graded ideal, it contains a non-zero homogeneous element. Then by using the elementary matrices, we can show that $J_1 = J_2$. So R contains a minimal graded right ideal, which we denote by I .

Since A is faithful, its annihilator is zero, so there is a homogeneous element $a \in A_\varepsilon$ for some $\varepsilon \in \Gamma$, such that $aI \neq 0$ (if not, then $aI = 0$ for all $a \in A^h$, so

$I \subseteq \text{Ann}(A) = 0$, contradicting $I \neq 0$). Then aI is a graded submodule of A as it is generated by the homogeneous elements $a \in A_\varepsilon$ and all the $i \in I^h$. But A is graded simple, so $aI = A$. Define a map

$$\begin{aligned} \psi : I &\longrightarrow (\varepsilon)aI = (\varepsilon)A \\ i &\longmapsto ai. \end{aligned}$$

This is a graded right R -module homomorphism, which is surjective. Since $\ker(\psi)$ is a graded right ideal of I , and I is minimal, this implies the kernel is zero, so ψ is a graded isomorphism. Similarly, we have $I \cong_{\text{gr}} (\varepsilon')B$ for some $\varepsilon' \in \Gamma$. So $(\varepsilon)A \cong_{\text{gr}} (\varepsilon')B$, which says $(\gamma)A \cong_{\text{gr}} B$ for some $\gamma \in \Gamma$. \square

Proposition 4.4.6. *Let D be a graded division ring, V be a non-zero graded D -module, and let $R = \text{End}_D(V)$. If $g : V \rightarrow V$ is a left shifted homomorphism of additive groups such that $gf = fg$ for all $f \in R$, then there is a homogeneous element $d \in D^h$ such that $g(v) = dv$ for all $v \in V$.*

Proof. Let $u \in V^h \setminus 0$. We will show u and $g(u)$ are linearly dependent over D . This is clear if $\dim_D(V) = 1$. Suppose $\dim_D(V) \geq 2$, and suppose they are linearly independent. Then $\{u, g(u)\}$ can be extended to form a homogeneous basis of V . We can define a map $f \in \text{End}_D(V)$ such that $f(u) = 0$ and $f(g(u)) = v \neq 0$ for some $v \in V$. We assumed $fg = gf$, so $f(g(u)) = g(f(u)) = g(0) = 0$, contradicting the choice of $f(g(u))$. So u and $g(u)$ are linearly dependent over D , and there is some $d \in D^h$ with $g(u) = du$. Let $v \in V$, and let $h \in R$ with $h(u) = v$. Then $g(v) = g(h(u)) = h(g(u)) = h(du) = d(h(u)) = dv$. \square

Proposition 4.4.7. *Let D, D' be graded division rings and let V, V' be graded modules of finite dimension n, n' over D, D' respectively. If $\text{End}_D(V) \cong_{\text{gr}} \text{End}_{D'}(V')$ as graded rings, then $\dim_D(V) = \dim_{D'}(V')$ and $D \cong_{\text{gr}} D'$.*

Proof. Let $R = \text{End}_D(V)$. Note that V is a graded right R -module via $vr = r(v)$ for all $v \in V, r \in R$. We will show that V is faithful and graded simple as a graded

right R -module. If $Vr = 0$ for $r \in R$, then $r(v) = 0$ for all $v \in V$, so $r = 0$. To show graded simple, let M be a non-zero graded right R -submodule of V , and let $m \in M^h \setminus 0$. Extend $\{m\}$ to form a homogeneous basis of V . For some $v \in V \setminus M$, define a map $\theta_v : V \rightarrow V$ by $\theta_v(m) = v$ and $\theta(w) = 0$ for all other elements w in this basis of V . Then $\theta_v \in R$, and $m\theta_v = v \notin M$, so M is not closed under action of R . This shows V is graded simple.

Denote the given graded ring isomorphism by $\sigma : R \rightarrow \text{End}_{D'}(V')$. Then using the same argument as above, we have that V' is a faithful and graded simple graded right $\text{End}_{D'}(V')$ -module. Using the isomorphism σ , we can give V' the structure of a graded right R -module by defining $wr = w\sigma(r)$ for each $w \in V'$, $r \in R$. It follows that V' is faithful with respect to R . Any graded right R -submodule of V' is also closed under action of $\text{End}_{D'}(V')$, so it is a graded right $\text{End}_{D'}(V')$ -submodule of V' . This shows that V' is graded simple as a graded right R -module.

Applying Proposition 4.4.5, there is a graded right R -module isomorphism $\phi : (\gamma)V \rightarrow V'$ for some $\gamma \in \Gamma$. Then for $v \in V$, $f \in R$,

$$(\phi \circ f)(v) = \phi(f(v)) = \phi(vf) = \phi(v)f = \phi(v)\sigma(f) = \sigma(f)(\phi(v)),$$

so that $\phi \circ f \circ \phi^{-1} = \sigma(f)$ as a homomorphism of additive groups from V' to V' . For $d \in D^h$, let $\alpha_d : V \rightarrow V$; $x \mapsto dx$. This is a (left shifted) homomorphism of additive groups. Clearly $\alpha_d = 0$ if and only if $d = 0$. Similarly for $e \in D'^h$ we can define α_e .

Let $f \in R$, $v \in V$ and $d \in D^h$. Then

$$(f \circ \alpha_d)(v) = f(dv) = df(v) = (\alpha_d \circ f)(v),$$

so $f \circ \alpha_d = \alpha_d \circ f$. Then using the above results $(\phi \circ \alpha_d \circ \phi^{-1}) \circ (\sigma(f)) = (\sigma(f)) \circ (\phi \circ \alpha_d \circ \phi^{-1})$. Since σ is surjective and $\phi \circ \alpha_d \circ \phi^{-1}$ is a left shifted homomorphism of additive groups from V' to V' , we can apply Proposition 4.4.6. There is a homogeneous element $e \in D'^h$ such that $\phi \circ \alpha_d \circ \phi^{-1}(w) = ew$ for all $w \in V'$.

Define a map

$$\begin{aligned}\tau : D^h &\longrightarrow D'^h \\ d &\longmapsto e\end{aligned}$$

and extend it linearly to cover all of D . It is routine to show that τ is a graded ring homomorphism. For example, we will show that, for some $\gamma \in \Gamma$, if $d_1, d_2 \in D_\gamma$ then $\tau(d_1 + d_2) = \tau(d_1) + \tau(d_2)$. The others parts are similar. Let $e_1, e_2, e_3 \in D'^h$ with $\phi \circ \alpha_{d_1} \circ \phi^{-1} = \alpha_{e_1}$, $\phi \circ \alpha_{d_2} \circ \phi^{-1} = \alpha_{e_2}$ and $\phi \circ \alpha_{d_1+d_2} \circ \phi^{-1} = \alpha_{e_3}$. Then for $w \in V'$,

$$\begin{aligned}e_3 w &= \phi \circ \alpha_{d_1+d_2} \circ \phi^{-1}(w) = \phi((d_1 + d_2)\phi^{-1}(w)) \\ &= \phi(d_1\phi^{-1}(w)) + \phi(d_2\phi^{-1}(w)) \\ &= \phi \circ \alpha_{d_1} \circ \phi^{-1}(w) + \phi \circ \alpha_{d_2} \circ \phi^{-1}(w) \\ &= e_1 w + e_2 w = (e_1 + e_2)w.\end{aligned}$$

Then τ is injective, since if $\tau(d) = 0$, then $\phi \circ \alpha_d \circ \phi^{-1}(w) = 0$ for all $w \in V'$. This implies $\alpha_d = 0$, so $d = 0$. Reverse the roles of D and D' and replace ϕ, σ by ϕ^{-1}, σ^{-1} . For each $k \in D'^h$, there is $d \in D^h$ such that $\phi^{-1} \circ \alpha_k \circ \phi = \alpha_d$. From above there is $\tau(d) \in D'^h$ such that $\phi \circ \alpha_d \circ \phi^{-1} = \alpha_{\tau(d)}$. Combining these gives $\alpha_k = \alpha_{\tau(d)}$. Since k and $\tau(d)$ are homogeneous of the same degree in D' , it follows that $k = \tau(d)$, so τ is surjective.

Let $d \in D^h$ and $v \in V$. Then $\phi(dv) = \phi \circ \alpha_d(v) = \alpha_{\tau(d)} \circ \phi(v) = \tau(d)\phi(v)$. Then using this we can show that $\{u_1, \dots, u_k\}$ are linearly independent in V if and only if $\{\phi(u_1), \dots, \phi(u_k)\}$ are linearly independent in V' . It follows that the former set spans V if and only if the latter set spans V' , proving $\dim_D(V) = \dim_{D'}(V')$. \square

Corollary 4.4.8. *Let D, D' be graded division rings, and let $(d) \in \Gamma^n$, $(d') \in \Gamma^{n'}$. If $M_n(D)(d) \cong_{\text{gr}} M_{n'}(D')(d')$ as graded rings, then $n = n'$ and $D \cong_{\text{gr}} D'$.*

Proof. As in the proof of Proposition 4.4.4, we can choose a graded D -module V such that $M_n(D)(d) \cong_{\text{gr}} \text{End}_D(V)$, and a graded D' -module V' such that $M_{n'}(D')(d') \cong_{\text{gr}}$

$\text{End}_{D'}(V')$. Then by Proposition 4.4.7, we have that $n = n'$ and $D \cong_{\text{gr}} D'$. \square

Let D be a Γ -graded division ring and let $(\lambda) = (\lambda_1, \dots, \lambda_n) \in \Gamma^n$. Consider the partition of Γ into right cosets of Γ_D . For distinct elements of (λ) , if the right cosets $\Gamma_D \lambda_1, \dots, \Gamma_D \lambda_n$ are not all distinct, then there is a first $\Gamma_D \lambda_i$ such that $\Gamma_D \lambda_i = \Gamma_D \lambda_j$ for some $j < i$. In (λ) , we will replace λ_i by λ_j , so that now $(\lambda) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_j, \lambda_{i+1}, \dots, \lambda_n)$. If the right cosets

$$\Gamma_D \lambda_1, \dots, \Gamma_D \lambda_{i-1}, \Gamma_D \lambda_{i+1}, \dots, \Gamma_D \lambda_n$$

are still not all distinct, repeat the above process. Continue repeating this process until all the right cosets of distinct elements of (λ) are distinct. Let k denote the number of distinct right cosets (which is, after the above process, also the number of distinct elements in (λ)). Let $\varepsilon_1 = \lambda_1$, let ε_2 be the second distinct element of (λ) , and so on until ε_k is the k -th distinct element of (λ) . For each ε_l , let r_l be the number of λ_i in $(\lambda_1, \dots, \lambda_n)$ with $\Gamma_D \lambda_i = \Gamma_D \varepsilon_l$. Using Proposition 4.4.4 we get

$$M_n(D)(\lambda_1, \dots, \lambda_n) \cong_{\text{gr}} M_n(D)(\varepsilon_1, \dots, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_2, \dots, \varepsilon_k, \dots, \varepsilon_k), \quad (4.2)$$

where each ε_l occurs r_l times.

Also note that for Γ -graded D -modules M and N , by Proposition 4.2.6, $\dim_D(M \oplus N) = \dim_D(M) + \dim_D((M \oplus N)/M)$. By the first isomorphism theorem, $(M \oplus N)/M \cong_{\text{gr}} N$, so $\dim_D(M \oplus N) = \dim_D(M) + \dim_D(N)$.

The following theorem extends [11, Thm. 2.1] from trivially graded fields to graded division rings.

Theorem 4.4.9. *Let D be a Γ -graded division ring, let $\lambda_i, \gamma_j \in \Gamma$ for $1 \leq i \leq n$, $1 \leq j \leq m$ and let $\Omega = \{\lambda_1, \dots, \lambda_n, \gamma_1, \dots, \gamma_m\}$. If the elements of Ω mutually commute and if*

$$M_n(D)(\lambda_1, \dots, \lambda_n) \cong_{\text{gr}} M_m(D)(\gamma_1, \dots, \gamma_m), \quad (4.3)$$

then we have $n = m$ and $\gamma_i = \tau_i \lambda_{\pi(i)} \sigma$ for each i , where $\tau_i \in \Gamma_D$, $\pi \in S_n$ is a permutation and $\sigma \in Z_\Gamma(\Omega)$ is fixed.

Proof. It follows from Corollary 4.4.8 that $n = m$. As in (4.2), we can find $(\epsilon) = (\epsilon_1, \dots, \epsilon_1, \epsilon_2, \dots, \epsilon_2, \dots, \epsilon_k, \dots, \epsilon_k)$ in Γ^n such that $M_n(D)(\lambda_1, \dots, \lambda_n) \cong_{\text{gr}} M_n(D)(\epsilon)$. Let $V = D(\epsilon_1^{-1}) \oplus \dots \oplus D(\epsilon_1^{-1}) \oplus \dots \oplus D(\epsilon_k^{-1}) \oplus \dots \oplus D(\epsilon_k^{-1})$, and let e_1, \dots, e_n be the standard homogeneous basis of V . Then $\deg(e_i) = \epsilon_{s_i}$, where ϵ_{s_i} is the i -th element in (ϵ) . Define $E_{ij} \in \text{End}_D(V)$ by $E_{ij}(e_l) = \delta_{il} e_j$, for $1 \leq i, j, l \leq n$. We observed above (before Proposition 4.4.4) that each E_{ij} is a graded D -module homomorphism of degree $\epsilon_{s_i}^{-1} \epsilon_{s_j}$, the set $\{E_{ij} : 1 \leq i, j \leq n\}$ forms a homogeneous basis for $\text{End}_D(V)$ and $\text{End}_D(V) \cong_{\text{gr}} M_n(D)(\epsilon)$, where E_{ij} corresponds to the matrix e_{ij} in $M_n(D)(\epsilon)$. For any i, j, h, l , we have $E_{ij} E_{hl} = \delta_{il} E_{hj}$, so $\{E_{ii} : 1 \leq i \leq n\}$ forms a complete system of orthogonal idempotents of $\text{End}_D(V)$.

Similarly, we can find $(\epsilon') = (\epsilon'_1, \dots, \epsilon'_1, \epsilon'_2, \dots, \epsilon'_2, \dots, \epsilon'_{k'}, \dots, \epsilon'_{k'}) \in \Gamma^n$ such that $M_n(D)(\gamma_1, \dots, \gamma_n) \cong_{\text{gr}} M_n(D)(\epsilon')$. Let

$$W = D(\epsilon'_1{}^{-1}) \oplus \dots \oplus D(\epsilon'_1{}^{-1}) \oplus \dots \oplus D(\epsilon'_{k'}{}^{-1}) \oplus \dots \oplus D(\epsilon'_{k'}{}^{-1})$$

and let e'_1, \dots, e'_n be the standard homogeneous basis of W with $\deg(e'_i) = \epsilon'_{s_i}$. We have $\text{End}_D(W) \cong_{\text{gr}} M_n(D)(\epsilon')$, so the graded isomorphism (4.3) provides a graded ring isomorphism $\theta : \text{End}_D(V) \rightarrow \text{End}_D(W)$. Define $E'_{ij} := \theta(E_{ij})$, for $1 \leq i, j \leq n$, and let $E'_{ii}(W) = Q_i$ for each i . Since $\{E_{ii} : 1 \leq i \leq n\}$ forms a complete system of orthogonal idempotents, $\{E'_{ii} : 1 \leq i \leq n\}$ also forms a complete system of orthogonal idempotents for $\text{End}_D(W)$. It follows that $W = \bigoplus_{1 \leq i \leq n} Q_i$.

For any i, j, h, l , as above, $E'_{ij} E'_{hl} = \delta_{il} E'_{hj}$ so $E'_{ij} E'_{ji} = E'_{jj}$ and E'_{ii} acts as the identity on Q_i . By restricting E'_{ij} to Q_i , these relations induce a graded D -module isomorphism $E'_{ij} : Q_i \rightarrow Q_j$ of the same degree as E_{ij} , namely $\epsilon_{s_i}^{-1} \epsilon_{s_j}$. So $Q_i \cong_{\text{gr}} Q_1(\epsilon_{s_i}^{-1} \epsilon_1)$ for any $1 \leq i \leq n$, and it follows that $W \cong_{\text{gr}} \bigoplus_{1 \leq i \leq n} Q_1(\epsilon_{s_i}^{-1} \epsilon_1)$. Using the observations before the theorem regarding dimension count, it follows that $\dim_D Q_1 = 1$. So Q_1 is generated by one homogeneous element, say q , with

$\deg(q) = \alpha$, and we have $Q_1 \cong_{\text{gr}} D(\alpha^{-1})$. Now $\Gamma_D \alpha = \text{Supp}(D(\alpha^{-1})) = \text{Supp}(Q_1) \subseteq \text{Supp}(W) = \bigcup_i (\Gamma_D \varepsilon'_i)$. But as the right cosets of Γ_D in Γ are either disjoint or equal, this implies $\Gamma_D \alpha = \Gamma_D \varepsilon'_j$ for some j . Then $Q_1 \cong_{\text{gr}} D(\alpha^{-1}) = D(\varepsilon'_j{}^{-1} \tau_j^{-1})$ for some $\tau_j \in \Gamma_D$. We can easily show that $D(\varepsilon'_j{}^{-1} \tau_j^{-1}) \cong_{\text{gr}} D(\varepsilon'_j{}^{-1})$, so $Q_1 \cong_{\text{gr}} D(\varepsilon'_j{}^{-1})$.

Thus $W \cong_{\text{gr}} \bigoplus_{1 \leq i \leq n} D(\varepsilon_{s_i}^{-1} \varepsilon_1 \varepsilon'_j{}^{-1})$. Then $V = \bigoplus_{1 \leq i \leq n} D(\varepsilon_{s_i}^{-1})$, so

$$V(\varepsilon_1 \varepsilon'_j{}^{-1}) \cong_{\text{gr}} \bigoplus_{1 \leq i \leq n} D(\varepsilon_{s_i}^{-1})(\varepsilon_1 \varepsilon'_j{}^{-1}) = \bigoplus_{1 \leq i \leq n} D(\varepsilon_1 \varepsilon'_j{}^{-1} \varepsilon_{s_i}^{-1}).$$

Using the assumption that the elements of Ω mutually commute, and since the elements of both (ε) and (ε') are elements of Ω , it follows that $\varepsilon_1 \varepsilon'_j{}^{-1} \varepsilon_{s_i}^{-1} = \varepsilon_{s_i}^{-1} \varepsilon_1 \varepsilon'_j{}^{-1}$. So we have $W \cong_{\text{gr}} V(\varepsilon_1 \varepsilon'_j{}^{-1})$. Let $\sigma = \varepsilon'_j \varepsilon_1^{-1}$ and denote this graded D -module isomorphism by $\phi : W \rightarrow V(\sigma^{-1})$.

Then $\phi(e'_i) = \sum_{1 \leq j \leq n} a_j e_j$, where $a_j \in D^h$ and e_j are homogeneous of degree $\varepsilon_{s_j} \sigma$ in $V(\sigma^{-1})$. Suppose that $\deg(e_j) \neq \deg(e_l)$ for some j, l with $a_j, a_l \neq 0$. Then since $\deg(\phi(e'_i)) = \varepsilon'_{s_i} = \deg(a_j e_j) = \deg(a_l e_l)$ and $\varepsilon_{s_j} \sigma \neq \varepsilon_{s_l} \sigma$, it follows that $\varepsilon_{s_j} \varepsilon_{s_l}^{-1} \in \Gamma_D$ which contradicts the fact that $\Gamma_D \varepsilon_{s_j}$ and $\Gamma_D \varepsilon_{s_l}$ are distinct. So for all non-zero terms in the sum, each e_j has the same degree. Thus $\varepsilon'_{s_i} = \tau_j \varepsilon_{s_j} \sigma$ where $\tau_j = \deg(a_j) \in \Gamma_D$. Each ε_{s_j} was chosen as $\varepsilon_{s_j} = \tau_l \lambda_l$ for some l with $\tau_l \in \Gamma_D$, and similarly each ε'_{s_i} was chosen as $\varepsilon'_{s_i} = \tau_h \gamma_h$ for some h with $\tau_h \in \Gamma_D$. Combining these gives that $\gamma_i = \tau_i \lambda_{\pi(i)} \sigma$ for $1 \leq i \leq n$, where $\tau_i \in \Gamma_D$ and $\sigma \in Z_\Gamma(\Omega)$. \square

Remark 4.4.10. We note that for the converse of the above theorem, we need to assume that $\sigma \in Z(\Gamma)$. Suppose $n = m$ and $\gamma_i = \tau_i \lambda_{\pi(i)} \sigma$ for each i , where $\tau_i \in \Gamma_D$, $\pi \in S_n$ is a permutation and $\sigma \in Z(\Gamma)$ is fixed. Then it follows immediately from Proposition 4.4.4 that

$$M_n(D)(\lambda_1, \dots, \lambda_n) \cong_{\text{gr}} M_m(D)(\gamma_1, \dots, \gamma_m).$$

Given a group Γ and a field K (which is not graded), it is also possible to define a grading on $M_n(K)$. Such a grading is defined to be a *good grading* of $M_n(K)$ if the matrices e_{ij} are homogeneous, where e_{ij} is the matrix with 1 in the ij -position.

These group gradings on matrix rings have been studied by Dăscălescu et al [15]. The following examples (from [15, Eg. 1.3]) show two examples of \mathbb{Z}_2 -grading on $M_2(K)$ for a field K , one of which is a good grading; the other is not a good grading.

Examples 4.4.11. 1. Let $R = M_2(K)$ with the \mathbb{Z}_2 grading defined by

$$R_{[0]} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in K \right\} \quad \text{and} \quad R_{[1]} = \left\{ \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} : c, d \in K \right\}$$

Then R is a graded ring with a good grading.

2. Let $S = M_2(K)$ with the \mathbb{Z}_2 grading defined by

$$S_{[0]} = \left\{ \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} : a, b \in K \right\} \quad \text{and} \quad S_{[1]} = \left\{ \begin{pmatrix} d & c \\ d & -d \end{pmatrix} : c, d \in K \right\}$$

Then S is a graded ring, such that the \mathbb{Z}_2 -grading is not a good grading, since e_{11} is not homogeneous. Note that R and S are graded isomorphic, as the map f is an isomorphism:

$$f : R \longrightarrow S; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a+c & b+d-a-c \\ c & d-c \end{pmatrix}$$

So S does not have a good grading, but it is isomorphic as a graded ring to R which has a good grading.

Let Γ be a group and let R be a Γ -graded ring. If V is an n -dimensional Γ -graded R -module then, as above (see §4.2 and page 76), $\text{END}_R(V) = \text{End}_R(V)$ forms a Γ -graded ring. Taking $R = K$ with the trivial Γ -grading, then $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$ is just a vector space with a Γ -grading, for subspaces V_γ of V .

Then $\text{End}_K(V) \cong M_n(K)$, which induces the structure of a Γ -graded K -algebra on $M_n(K)$. Suppose $\{b_1, \dots, b_n\}$ forms a homogeneous basis for V , with $\deg(b_i) = \gamma_i$. Then the matrix $e_{ij} \in M_n(K)$ corresponds to the map $E_{ij} \in \text{End}_K(V)$ defined by $E_{ij}(b_l) = \delta_{il}b_j$, where δ is the Kronecker delta. As E_{ij} is homogeneous of degree

$\gamma_i^{-1}\gamma_j$, e_{ij} is also homogeneous, so this construction gives a good Γ -grading on $M_n(K)$. This observation, combined with the following proposition, shows that a grading is good if and only if it can be obtained in this manner from an n -dimensional Γ -graded K -vector space.

Proposition 4.4.12. *Consider a good Γ -grading on $M_n(K)$. Then there exists a Γ -graded vector space V such that the isomorphism $\text{End}(V) \cong M_n(K)$, with respect to a homogeneous basis of V , is an isomorphism of Γ -graded algebras.*

Proof. See [15, Prop. 1.2]. □

4.5 Graded projective modules

Let Γ be a group which is not necessarily abelian and R be a Γ -graded ring. Throughout this section, unless otherwise stated, we will assume that all graded rings, graded modules and graded algebras are also Γ -graded.

We will consider the category $R\text{-gr-Mod}$ which is defined as follows: the objects are Γ -graded (left) R -modules, and for two objects M, N in $R\text{-gr-Mod}$, the morphisms are defined as

$$\text{Hom}_{R\text{-gr-Mod}}(M, N) = \{f \in \text{Hom}_R(M, N) : f(M_\gamma) \subseteq N_\gamma \text{ for all } \gamma \in \Gamma\}.$$

A graded R -module $P = \bigoplus_{\gamma \in \Gamma} P_\gamma$ is said to be *graded projective* (resp. *graded faithfully projective*) if P is projective (resp. faithfully projective) as an R -module. We use $\mathcal{Pgr}(R)$ to denote the subcategory of $R\text{-gr-Mod}$ consisting of graded finitely generated projective modules over R . Then Proposition 4.5.2 shows some equivalent characterisations of graded projective modules. Its proof requires the following proposition, which is from [51, Prop. 2.3.1].

Proposition 4.5.1. *Let L, M, N be graded R -modules, with R -linear maps*

$$\begin{array}{ccc} & M & \\ h \nearrow & & \searrow g \\ L & \xrightarrow{f} & N \end{array}$$

such that $f = g \circ h$ and f is a morphism in the category $R\text{-gr-Mod}$. If g (resp. h) is a morphism in $R\text{-gr-Mod}$, then there exists a morphism $h' : L \rightarrow M$ (resp. $g' : M \rightarrow N$) in $R\text{-gr-Mod}$ such that $f = g \circ h'$ (resp. $f = g' \circ h$).

Proof. See [51, Prop. 2.3.1]. □

Theorem 4.5.2. *Let R be a Γ -graded ring and let P be a graded R -module. Then the following are equivalent:*

1. P is graded projective;
2. For each diagram of graded R -module homomorphisms

$$\begin{array}{ccc} & P & \\ & \downarrow j & \\ M & \xrightarrow{g} & N \longrightarrow 0 \end{array}$$

with g surjective, there is a graded R -module homomorphism $h : P \rightarrow M$ with $g \circ h = j$;

3. $\text{Hom}_{R\text{-gr-Mod}}(P, -)$ is an exact functor in $R\text{-gr-Mod}$;
4. Every short exact sequence of graded R -module homomorphisms

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

splits via a graded map;

5. P is graded isomorphic to a direct summand of a graded free R -module F .

Proof. (1) \Rightarrow (2): Since P is projective and g is a surjective R -module homomorphism, there is an R -module homomorphism $h : P \rightarrow M$ with $g \circ h = j$. By Proposition 4.5.1, there is a graded R -module homomorphism $h' : P \rightarrow M$ with $g \circ h' = j$.

(2) \Rightarrow (3): In exactly the same way as the non-graded setting (see [37, Thm. IV.4.2]) we can show that $\text{Hom}_{R\text{-gr-Mod}}(P, -)$ is left exact. Then it follows immediately from (2) that it is right exact.

(3) \Rightarrow (4): This is immediate.

(4) \Rightarrow (5): Let $\{p_i\}_{i \in I}$ be a homogeneous generating set for P , where $\deg(p_i) = \delta_i$. Let $\bigoplus_{i \in I} R(\delta_i^{-1})$ be the graded free R -module with standard homogeneous basis $\{e_i\}_{i \in I}$ where $\deg(e_i) = \delta_i$. Then there is an exact sequence

$$0 \longrightarrow \ker(g) \xrightarrow{\subseteq} \bigoplus_{i \in I} R(\delta_i^{-1}) \xrightarrow{g} P \longrightarrow 0,$$

since the map $g : \bigoplus_{i \in I} R(\delta_i^{-1}) \rightarrow P$; $e_i \mapsto p_i$ is a surjective graded R -module homomorphism. By (4), there is a graded R -module homomorphism $h : P \rightarrow \bigoplus_{i \in I} R(\delta_i^{-1})$ such that $g \circ h = \text{id}_P$.

Since the exact sequence is, in particular, a split exact sequence of R -modules, we know from the non-graded setting [48, Prop. 2.5] that there is an R -module isomorphism

$$\begin{aligned} \theta : P \oplus \ker(g) &\longrightarrow \bigoplus_{i \in I} R(\delta_i^{-1}) \\ (p, q) &\longmapsto h(p) + q. \end{aligned}$$

Clearly this map is also a graded R -module homomorphism, so $P \oplus \ker(g) \cong_{\text{gr}} \bigoplus_{i \in I} R(\delta_i^{-1})$.

(5) \Rightarrow (1): Graded free modules are free, so P is isomorphic to a direct summand of a free R -module. From the non-graded setting, we know that P is projective. \square

Let Γ be an abelian group, R be a commutative Γ -graded ring, and we define

multiplication in $\text{End}_R(A)$ to be $g \cdot f = g \circ f$.

Definition 4.5.3. A graded R -algebra A is called a *graded Azumaya algebra* if the following two conditions hold:

1. A is graded faithfully projective;
2. The natural map $\psi_A : A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A)$ is a graded isomorphism.

We note that a graded R -algebra which is an Azumaya algebra (in the non-graded sense) is also a graded Azumaya algebra, since it is faithfully projective as an R -module, and the natural homomorphism $A \otimes_R A^{\text{op}} \rightarrow \text{End}_R(A)$ is clearly graded. So a graded central simple algebra over a graded field (as in Theorem 4.3.3) is in fact a graded Azumaya algebra.

The following proposition proves, in the graded setting, a partial result of Morita equivalence (only in one direction), which we will use in the next chapter (see Proposition 5.4.3). For the following proposition, we require the group Γ to be an abelian group, so it will be written additively. We observe that if Γ is an abelian group, with R a graded ring and $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$, then $R^n(d)$ is a graded $M_n(R)(d)$ - R -bimodule and $R^n(-d)$ is a graded R - $M_n(R)(d)$ -bimodule. By [51, p. 30], we can define grading on the tensor product of two graded modules in a similar way to that of graded algebras (as we defined on page 76).

Proposition 4.5.4 (Morita Equivalence in the graded setting). *Let Γ be an abelian group, R be a graded ring and let $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$. Then the functors*

$$\begin{aligned} \psi : \mathcal{P}\text{gr}(M_n(R)(d)) &\longrightarrow \mathcal{P}\text{gr}(R) \\ P &\longmapsto R^n(-d) \otimes_{M_n(R)(d)} P \\ \text{and} \quad \varphi : \mathcal{P}\text{gr}(R) &\longrightarrow \mathcal{P}\text{gr}(M_n(R)(d)) \\ Q &\longmapsto R^n(d) \otimes_R Q \end{aligned}$$

form equivalences of categories.

Proof. There are graded R -module homomorphisms

$$\begin{aligned}\theta : R^n(-d) \otimes_{M_n(R)(d)} R^n(d) &\longrightarrow R \\ (a_1, \dots, a_n) \otimes (b_1, \dots, b_n) &\longmapsto a_1 b_1 + \dots + a_n b_n; \\ \text{and } \sigma : R &\longrightarrow R^n(-d) \otimes_{M_n(R)(d)} R^n(d) \\ a &\longmapsto (a, 0, \dots, 0) \otimes (1, 0, \dots, 0)\end{aligned}$$

with $\sigma \circ \theta = \text{id}$ and $\theta \circ \sigma = \text{id}$. Further

$$\begin{aligned}\theta' : R^n(d) \otimes_R R^n(-d) &\longrightarrow M_n(R)(d) \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} &\longmapsto \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & & \vdots \\ a_n b_1 & \cdots & a_n b_n \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{and } \sigma' : M_n(R)(d) &\longrightarrow R^n(d) \otimes_R R^n(-d) \\ (m_{i,j}) &\longmapsto \begin{pmatrix} m_{1,1} \\ m_{2,1} \\ \vdots \\ m_{n,1} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + \begin{pmatrix} m_{1,n} \\ \vdots \\ m_{n-1,n} \\ m_{n,n} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\end{aligned}$$

are graded $M_n(R)(d)$ -module homomorphisms with $\sigma' \circ \theta' = \text{id}$ and $\theta' \circ \sigma' = \text{id}$. So $R^n(-d) \otimes_{M_n(R)(d)} R^n(d) \cong_{\text{gr}} R$ as graded R - R -bimodules and $R^n(d) \otimes_R R^n(-d) \cong_{\text{gr}} M_n(R)(d)$ as graded $M_n(R)(d)$ - $M_n(R)(d)$ -bimodules respectively.

Then for $P \in \mathcal{P}\text{gr}(M_n(R)(d))$, $R^n(d) \otimes_R R^n(-d) \otimes_{M_n(R)(d)} P \cong_{\text{gr}} P$. Suppose $f : P \rightarrow P'$ is a graded $M_n(R)(d)$ -module homomorphism. Then there is a commutative diagram

$$\begin{array}{ccc} R^n(d) \otimes_R R^n(-d) \otimes_{M_n(R)(d)} P & \longrightarrow & P \\ \text{id} \otimes f \downarrow & & \downarrow f \\ R^n(d) \otimes_R R^n(-d) \otimes_{M_n(R)(d)} P' & \longrightarrow & P' \end{array}$$

For $Q \in \mathcal{P}\text{gr}(R)$, $R^n(-d) \otimes_{M_n(R)(d)} R^n(d) \otimes_R Q \cong_{\text{gr}} Q$. If $g : Q \rightarrow Q'$ is a graded R -module homomorphism, then there is a commutative diagram

$$\begin{array}{ccc} R^n(-d) \otimes_{M_n(R)(d)} R^n(d) \otimes_R Q & \longrightarrow & Q \\ \text{id} \otimes g \downarrow & & \downarrow g \\ R^n(-d) \otimes_{M_n(R)(d)} R^n(d) \otimes_R Q' & \longrightarrow & Q' \end{array}$$

This shows that there are natural equivalences from $\psi \circ \varphi$ to the identity functor and from $\varphi \circ \psi$ to the identity. Hence ψ and φ are mutually inverse equivalences of categories. \square

Chapter 5

Graded K -Theory of Azumaya Algebras

In Corollary 4.3.4, we saw that the K -theory of a graded central simple algebra, graded by an abelian group, is very close to the K -theory of its centre, where this follows immediately from the corresponding result in the non-graded setting (see Theorem 3.1.5). Note that for the K -theory of a graded central simple algebra A , we are considering $K_i(A) = K_i(\mathcal{P}\mathbf{r}(A))$, where $\mathcal{P}\mathbf{r}(A)$ denotes the category of finitely generated projective A -modules. But in the graded setting, there is also the category of graded finitely generated projective modules over a given graded ring, which we will consider in Section 5.4.

In Section 5.1, we consider the functor K_0 in the graded setting. For a graded ring R , we set $K_0^{\text{gr}}(R) = K_0(\mathcal{P}\mathbf{gr}(R))$, where $\mathcal{P}\mathbf{gr}(R)$ is the category of graded finitely generated projective R -modules. We show that this definition is equivalent to defining $K_0^{\text{gr}}(R)$ to be the group completion of $\mathcal{P}\mathbf{roj}^{\text{gr}}(R)$, where $\mathcal{P}\mathbf{roj}^{\text{gr}}(R)$ denotes the monoid of isomorphism classes of graded finitely generated projective modules over R .

We include a number of results involving K_0 of strongly graded rings in Section 5.2. Then in Section 5.3, we consider a specific example of a graded Azumaya algebra to show that the graded K -theory of this graded Azumaya algebra is not the

same as the graded K -theory of its centre (see Example 5.3.2). So for this example, in the setting of graded K -theory, Corollary 4.3.4 does not hold. We also provide in Section 5.3 an example of a graded Azumaya algebra such that its (non-graded) K -theory does not coincide with its graded K -theory.

In Section 5.4, we study the graded K -theory of graded Azumaya algebras. We introduce an abstract functor called a graded \mathcal{D} -functor defined on the category of graded Azumaya algebras graded free over a fixed commutative graded ring R (Definition 5.4.1). As in Section 3.1, this allows us to show that, for a graded Azumaya algebra A graded free over R and subject to certain conditions, we have a relation similar to (3.1) in the graded setting (see Theorem 5.4.4 and [32]). A corollary of this is that the isomorphism holds for any Azumaya algebra free over its centre (see Corollary 5.4.7 and Theorem 3.1.5).

5.1 Graded K_0

In Section 2.1, we defined K_0 of a ring R to be the group completion of the monoid $\text{Proj}(R)$ of isomorphism classes of finitely generated projective R -modules. In [58, Ch. 3], there is another definition of K_0 , which defines K_0 for categories, rather than just for rings. Then using this definition of K_0 for categories, K_0 of a ring R can also be defined to be $K_0(\text{Pr}(R))$, which is shown in [58, Thm. 3.1.7] to be equivalent to the definition of K_0 given in Section 2.1.

In this section, we let Γ be a multiplicative group and R be a Γ -graded ring. We begin this section by stating the definition of a category with exact sequences. We then show that $\mathcal{Pgr}(R)$ is a category with exact sequences, and we recall from [58, Defn. 3.1.6] the definition of K_0 of a category (see Definition 5.1.2).

Definition 5.1.1. A *category with exact sequences* is a full additive subcategory \mathcal{C} of an abelian category \mathcal{A} , with the following properties:

1. \mathcal{C} is closed under extensions; that is, if

$$0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$$

is an exact sequence in \mathcal{A} and $P_1, P_2 \in \text{Obj } \mathcal{C}$, then $P \in \text{Obj } \mathcal{C}$.

2. \mathcal{C} has a small skeleton; that is, \mathcal{C} has a full subcategory \mathcal{C}_0 such that $\text{Obj } \mathcal{C}_0$ is a set, and for which the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is an equivalence.

The exact sequences in such a category are defined to be the exact sequences in the ambient category \mathcal{A} involving only objects (and morphisms) all chosen from \mathcal{C} .

Let R be a graded ring. We will show that $\mathcal{Pgr}(R)$ is a category with exact sequences. Firstly note that $\mathcal{Pgr}(R)$ is a full additive subcategory of $R\text{-gr-Mod}$, where $R\text{-gr-Mod}$ is an abelian category (by [51, §2.2]). Take

$$0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$$

to be an exact sequence in $R\text{-gr-Mod}$. If $P_1, P_2 \in \text{Obj } \mathcal{Pgr}(R)$, then $P_1 \oplus Q_1 \cong_{\text{gr}} R^n(\delta_1, \dots, \delta_n)$ and $P_2 \oplus Q_2 \cong_{\text{gr}} R^m(\delta_{n+1}, \dots, \delta_{n+m})$, for some $\delta_i \in \Gamma$. So by Theorem 4.5.2, $P \cong_{\text{gr}} P_1 \oplus P_2$ from the exact sequence and

$$P \oplus (Q_1 \oplus Q_2) \cong_{\text{gr}} (P_1 \oplus Q_1) \oplus (P_2 \oplus Q_2) \cong_{\text{gr}} R^{n+m}(\delta_1, \dots, \delta_{n+m});$$

that is, $P \in \text{Obj } \mathcal{Pgr}(R)$.

For the second property, take \mathcal{C}_0 to be the set of graded direct summands of $\{R^n(d) : n \in \mathbb{N}, (d) \in \Gamma^n\}$. We will observe here that this is a set. Graded direct summands are, in particular, direct summands, and we know from the non-graded setting that the collection of direct summands of $\{R^n : n \in \mathbb{N}\}$ is a set. Taking the union of these sets over all $(d) \in \Gamma^n$, we still have a set since the union of sets indexed by a set is still a set.

For the second part of property (2), we have an equivalence of categories from \mathcal{C}_0 to $\mathcal{Pgr}(R)$. This follows since for any $P \in \mathcal{Pgr}(R)$, we know from Theorem 4.5.2 that

P is graded isomorphic to a graded direct summand of $\{R^n(d) : n \in \mathbb{N}, (d) \in \Gamma^n\}$. This gives a functor $\mathcal{F} : \mathcal{Pgr}(R) \rightarrow \mathcal{C}_0$. Consider the inclusion functor $\mathcal{J} : \mathcal{C}_0 \rightarrow \mathcal{Pgr}(R)$. Then it is routine to check that the functors \mathcal{F} and \mathcal{J} form mutually inverse equivalences of categories.

Definition 5.1.2. Let \mathcal{C} be a category with exact sequences with small skeleton \mathcal{C}_0 . Then $K_0(\mathcal{C})$ is defined to be the free abelian group based on $\text{Obj } \mathcal{C}_0$, modulo the following relations:

1. $[P] = [P']$ if there is an isomorphism $P \cong P'$ in \mathcal{C} .
2. $[P] = [P_1] + [P_2]$ if there is a short exact sequence $0 \rightarrow P_1 \rightarrow P \rightarrow P_2 \rightarrow 0$ in \mathcal{C} .

Here $[P]$ denotes the element of $K_0(\mathcal{C})$ corresponding to $P \in \text{Obj } \mathcal{C}_0$. We note that relation (1) is a special case of (2) with $P_1 = 0$. Since every $P \in \text{Obj } \mathcal{C}$ is isomorphic to an object of \mathcal{C}_0 , the notation $[P]$ makes sense for every object of \mathcal{C} .

We will now observe that $\text{Proj}^{\text{gr}}(R)$, the isomorphism classes of graded finitely generated projective modules, forms a monoid. Firstly it is a set, by the above argument, since it is the set of direct summands of $\{R^n(d) : n \in \mathbb{N}, (d) \in \Gamma^n\}$, modulo the equivalence relation of graded isomorphism. We will show the direct sum on $\text{Proj}^{\text{gr}}(R)$ is well defined. If $[P] = [P']$ and $[Q] = [Q']$, then $P \cong_{\text{gr}} P'$ and $Q \cong_{\text{gr}} Q'$. So we have $P \oplus Q \cong_{\text{gr}} P' \oplus Q'$; that is, $[P] + [Q] = [P'] + [Q']$. Clearly the binary operation is commutative and associative, and the identity element in $\text{Proj}^{\text{gr}}(R)$ is the isomorphism class of the zero module.

For a graded ring R , we define

$$K_0^{\text{gr}}(R) = K_0(\text{Proj}^{\text{gr}}(R)),$$

where $K_0(\text{Proj}^{\text{gr}}(R))$ is defined to be K_0 of the category $\text{Proj}^{\text{gr}}(R)$. We show in Theorem 5.1.3 that this definition of K_0^{gr} is equivalent to defining $K_0^{\text{gr}}(R)$ to be the group completion of $\text{Proj}^{\text{gr}}(R)$ (see [58, Thm. 1.1.3] for the group completion construction).

The following proof follows in a similar way to the equivalent result in the non-graded setting (see [58, Thm. 3.1.7]).

Theorem 5.1.3. *Let R be a graded ring and let $\mathcal{P}\text{gr}(R)$ be the category of graded finitely generated projective modules over R . Then the group completion of $\text{Proj}^{\text{gr}}(R)$ may be identified naturally with $K_0(\mathcal{P}\text{gr}(R))$, where $K_0(\mathcal{P}\text{gr}(R))$ is defined in Definition 5.1.2.*

Proof. By definition, $K_0(\mathcal{P}\text{gr}(R))$ and the group completion of $\text{Proj}^{\text{gr}}(R)$ are both defined to be abelian groups with one generator for each isomorphism class of graded finitely generated projective modules over R .

Addition in the latter is defined as $[P] + [Q] = [P \oplus Q]$. In $K_0(\mathcal{P}\text{gr}(R))$, $[P] + [Q]$ is defined to be $[N]$ for a graded finitely generated projective R -module N if there is an exact sequence

$$0 \longrightarrow P \longrightarrow N \longrightarrow Q \longrightarrow 0$$

in $\mathcal{P}\text{gr}(R)$. If $N = P \oplus Q$ then there is clearly an exact sequence

$$0 \longrightarrow P \longrightarrow P \oplus Q \longrightarrow Q \longrightarrow 0$$

in $\mathcal{P}\text{gr}(R)$. Thus $[P \oplus Q] = [N]$, and the addition operations in the two groups coincide.

We also need to check that both groups satisfy the same relations. The group completion construction is the free abelian group based on $\text{Proj}^{\text{gr}}(R)$ subject to the relations $[P] = [P']$ if $P \cong_{\text{gr}} P'$ and $[P] + [Q] = [P \oplus Q]$. We have observed above that the second relation holds in $K_0(\mathcal{P}\text{gr}(R))$, and it is clear that the first relation also holds.

Then $K_0(\mathcal{P}\text{gr}(R))$ is subject to the relations (1) and (2) as in Definition 5.1.2. It remains to check that the second of these relations holds in the group completion construction. Suppose

$$0 \longrightarrow P_1 \xrightarrow{f} P \xrightarrow{g} P_2 \longrightarrow 0$$

is an exact sequence in $\mathcal{P}\text{gr}(R)$. By Theorem 4.5.2 this is split exact since P_2 is graded projective. So there is a graded R -module homomorphism $h : P_2 \rightarrow P$ with $g \circ h = \text{id}_{P_2}$. Then as in the proof of Theorem 4.5.2, there is a graded R -module isomorphism

$$\begin{aligned} P_1 \oplus P_2 &\longrightarrow P \\ (p, q) &\longmapsto f(p) + h(q). \end{aligned}$$

So with the group completion construction, $[P] = [P_1 \oplus P_2] = [P_1] + [P_2]$ which shows that it satisfies the second relation. \square

We also observe that by [58, p. 291], for $i = 0$, Quillen's Q -construction of K_0 for a category (as in Section 2.3) coincides with Definition 5.1.2 of K_0 of a category. We finish this section by calculating K_0^{gr} of a trivially graded field.

Proposition 5.1.4. *Let F be a field, Γ be a group and consider F as a trivially Γ -graded field. Then $K_0^{\text{gr}}(F) \cong \bigoplus_{\gamma \in \Gamma} \mathbb{Z}_{\gamma}$ where $\mathbb{Z}_{\gamma} = \mathbb{Z}$ for each $\gamma \in \Gamma$.*

Proof. Let M be a graded finitely generated projective F -module. By Theorem 4.2.4, M is graded free so $M \cong_{\text{gr}} F(\delta_1)^{r_1} \oplus \cdots \oplus F(\delta_k)^{r_k}$, where $r_i \in \mathbb{N}$ and the $\delta_i \in \Gamma$ are distinct. To show that M is written uniquely in this way, suppose $M \cong_{\text{gr}} F(\alpha_1)^{r'_1} \oplus \cdots \oplus F(\alpha_l)^{r'_l}$ for some $r'_i \in \mathbb{N}$ and some distinct $\alpha_i \in \Gamma$. Consider the set $\{\gamma_1, \dots, \gamma_n\} = \{\delta_1, \dots, \delta_k\} \cup \{\alpha_1, \dots, \alpha_l\}$. By rearranging the terms and adding zeros where required, we have

$$F(\delta_1)^{r_1} \oplus \cdots \oplus F(\delta_k)^{r_k} = F(\gamma_1)^{s_1} \oplus \cdots \oplus F(\gamma_n)^{s_n}$$

where $s_i \in \mathbb{N}$ and the γ_i are distinct. Similarly, $F(\alpha_1)^{r'_1} \oplus \cdots \oplus F(\alpha_l)^{r'_l}$ can be written as $F(\gamma_1)^{s'_1} \oplus \cdots \oplus F(\gamma_n)^{s'_n}$.

We note that as F_e -modules

$$(F(\gamma_1)^{s_1} \oplus \cdots \oplus F(\gamma_n)^{s_n})_{\gamma_1^{-1}} \cong (F(\gamma_1)^{s'_1} \oplus \cdots \oplus F(\gamma_n)^{s'_n})_{\gamma_1^{-1}}.$$

If $s_1 = 0$, then on the left hand side of the isomorphism we have zero, since $F(\gamma_i)_{\gamma_i^{-1}} = 0$ for all $i \neq 1$. So we must also have zero on the right hand side of the isomorphism. Therefore as $F(\gamma_1)_{\gamma_1^{-1}} = F_e = F$, we must have $s'_1 = 0$. If $s_1 \neq 0$, then on the left hand side of the isomorphism, we have F^{s_1} . Since it is an F -module isomorphism, we have the same on the right hand side, and thus $s_1 = s'_1$.

Repeat the same argument for each $\gamma_i \in \{\gamma_1, \dots, \gamma_n\}$. This shows that for each i , we have $s_i = s'_i$. Thus M can be written uniquely as $F(\delta_1)^{r_1} \oplus \dots \oplus F(\delta_k)^{r_k}$, where $r_i \in \mathbb{N}$ and the $\delta_i \in \Gamma$ are distinct. This gives an isomorphism from $\text{Proj}^{\text{gr}}(F)$ to $\bigoplus_{\gamma \in \Gamma} \mathbb{N}_{\gamma}$ where $\mathbb{N}_{\gamma} = \mathbb{N}$ for each $\gamma \in \Gamma$. As the group completion of \mathbb{N} is \mathbb{Z} , it follows that $K_0^{\text{gr}}(F)$ is isomorphic to $\bigoplus_{\gamma \in \Gamma} \mathbb{Z}_{\gamma}$ where $\mathbb{Z}_{\gamma} = \mathbb{Z}$ for each $\gamma \in \Gamma$. \square

5.2 Graded K_0 of strongly graded rings

Throughout this section, we let Γ be a multiplicative group and R be a Γ -graded ring. For any R_e -module N and any $\gamma \in \Gamma$, we identify the R_e -module $R_{\gamma} \otimes_{R_e} N$ with its image in $R \otimes_{R_e} N$. Then $R \otimes_{R_e} N$ is a Γ -graded R -module, with $R \otimes_{R_e} N = \bigoplus_{\gamma \in \Gamma} R_{\gamma} \otimes_{R_e} N$. Consider the restriction functor

$$\mathcal{G} : R\text{-gr-Mod} \longrightarrow R_e\text{-Mod}$$

$$M \longmapsto M_e$$

$$\psi \longmapsto \psi|_{M_e},$$

and the induction functor defined by

$$\mathcal{I} : R_e\text{-Mod} \longrightarrow R\text{-gr-Mod}$$

$$N \longmapsto R \otimes_{R_e} N$$

$$\phi \longmapsto \text{id}_R \otimes \phi.$$

Proposition 5.2.1 and Theorem 5.2.2 are from Dade [14, p. 245] (see also [51, Thm. 3.1.1]).

Proposition 5.2.1. *Let R be a Γ -graded ring. With \mathcal{G} and \mathcal{I} defined as above, there is a natural equivalence of the composite functor $\mathcal{G} \circ \mathcal{I} : R_e\text{-Mod} \rightarrow R_e\text{-Mod}$ with the identity functor on $R_e\text{-Mod}$.*

Proof. Let $N \in R_e\text{-Mod}$. Then $\mathcal{G} \circ \mathcal{I}(N) = (R \otimes_{R_e} N)_e = R_e \otimes_{R_e} N$. We know that for the ring R_e , the map $\alpha : R_e \otimes_{R_e} N \rightarrow N; r \otimes n \mapsto rn$ is an isomorphism. For $\phi : N \rightarrow N'$ in $R_e\text{-Mod}$, the following diagram is clearly commutative

$$\begin{array}{ccc} R_e \otimes_{R_e} N & \xrightarrow{\alpha} & N \\ \text{id}_{R_e} \otimes \phi \downarrow & & \downarrow \phi \\ R_e \otimes_{R_e} N' & \xrightarrow{\alpha} & N' \end{array}$$

So α is a natural equivalence from $\mathcal{G} \circ \mathcal{I}$ to the identity functor. \square

Theorem 5.2.2 (Dade's Theorem). *Let R be a Γ -graded ring. If R is strongly graded, then the functors \mathcal{G} and \mathcal{I} defined above form mutually inverse equivalences of categories.*

Proof. Let M be a graded R -module. Then $\mathcal{I} \circ \mathcal{G}(M) = R \otimes_{R_e} M_e$. We will show that the natural map $\beta : R \otimes_{R_e} M_e \rightarrow M; r \otimes m \mapsto rm$ is a graded R -module isomorphism. It is an R -module homomorphism using properties of tensor products, and is clearly graded. Since R is strongly graded, it follows that for all $\gamma, \delta \in \Gamma$,

$$M_{\gamma\delta} = R_e M_{\gamma\delta} = R_\gamma R_{\gamma^{-1}} M_{\gamma\delta} \subseteq R_\gamma M_\delta \subseteq M_{\gamma\delta}$$

so we have $R_\gamma M_\delta = M_{\gamma\delta}$. We note that $\beta(R_\gamma \otimes_{R_e} M_e) = R_\gamma M_e = M_\gamma$, so β is surjective.

Let $N = \ker(\beta)$, which is a graded R -submodule of $R \otimes_{R_e} M_e$, so $N_e = N \cap (R_e \otimes_{R_e} M_e)$. Now $N_e = \ker(\alpha)$, where $\alpha : R_e \otimes_{R_e} M_e \rightarrow M_e$ is the canonical isomorphism, so $N_e = 0$. Since N is a graded R -module, as above we have $N_\gamma = R_\gamma N_e = 0$ for all $\gamma \in \Gamma$. It follows that β is injective.

Let $\psi : M \rightarrow M'$ in $R\text{-gr-Mod}$. Then the following diagram commutes

$$\begin{array}{ccc} R \otimes_{R_e} M_e & \xrightarrow{\beta} & M \\ \text{id}_R \otimes \psi|_{M_e} \downarrow & & \downarrow \psi \\ R \otimes_{R_e} M'_e & \xrightarrow{\beta} & M' \end{array}$$

so β is a natural equivalence from $\mathcal{J} \circ \mathcal{G}$ to the identity functor, completing the proof.

□

Proposition 5.2.3. *Let R be a Γ -graded ring. If R is strongly graded, then for each $\gamma \in \Gamma$, R_γ is a finitely generated projective left (or right) R_e -module.*

Proof. See [50, Cor. 2.16.10].

□

We now show that the functors \mathcal{G} and \mathcal{J} , when restricted to the categories of finitely generated projective modules, still form an equivalence of categories.

Corollary 5.2.4. *Let R be a Γ -graded ring. If R is strongly graded, then the functors*

$$\mathcal{G} : \mathcal{Pgr}(R) \longrightarrow \mathcal{Pr}(R_e) \quad \text{and} \quad \mathcal{J} : \mathcal{Pr}(R_e) \longrightarrow \mathcal{Pgr}(R)$$

form mutually inverse equivalences of categories.

Proof. If $A \in \mathcal{Pr}(R_e)$, then $A \oplus B \cong R_e^n$ for some R_e -module B . Then $\mathcal{J}(A \oplus B) \cong \mathcal{J}(R_e^n)$, so $\mathcal{J}(A) \oplus \mathcal{J}(B) \cong R^n$. This shows $\mathcal{J}(A)$ is finitely generated and projective as an R -module, and we know $\mathcal{J}(A) \in R\text{-gr-Mod}$, so $\mathcal{J}(A) \in \mathcal{Pgr}(R)$.

If $M \in \mathcal{Pgr}(R)$, then $M \oplus N \cong_{\text{gr}} R^n(d)$ for some graded R -module N and some $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$. Then $\mathcal{G}(M \oplus N) \cong \mathcal{G}(R^n(d))$, so $\mathcal{G}(M) \oplus \mathcal{G}(N) \cong R_{\delta_1} \oplus \dots \oplus R_{\delta_n}$. Since, by Proposition 5.2.3, each R_{δ_i} is a finitely generated projective module over R_e , we have that $\mathcal{G}(M)$ is also finitely generated and projective as an R_e -module. The result now follows from Theorem 5.2.2. □

We note that \mathcal{J} and \mathcal{G} as in Corollary 5.2.4 are exact functors between exact categories. This follows since if $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence in

$\mathcal{P}\text{gr}(R)$, then as N is graded projective, the exact sequence splits. So $M \cong_{\text{gr}} L \oplus N$ and $\mathcal{G}(M) \cong \mathcal{G}(L \oplus N) \cong \mathcal{G}(L) \oplus \mathcal{G}(N)$. Thus $0 \rightarrow \mathcal{G}(L) \rightarrow \mathcal{G}(M) \rightarrow \mathcal{G}(N) \rightarrow 0$ is a split exact sequence in $\mathcal{P}\text{r}(R_e)$.

Suppose $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence in $\mathcal{P}\text{r}(R_e)$. Then the exact sequence splits, so $B \cong A \oplus C$ and $\mathcal{J}(B) \cong_{\text{gr}} \mathcal{J}(A \oplus C) \cong_{\text{gr}} \mathcal{J}(A) \oplus \mathcal{J}(C)$. Define R -module homomorphisms

$$\begin{aligned} \pi : \mathcal{J}(A) \oplus \mathcal{J}(C) &\longrightarrow \mathcal{J}(C) & \text{and} & & \iota : \mathcal{J}(A) &\longrightarrow \mathcal{J}(A) \oplus \mathcal{J}(C) \\ (a, c) &\longmapsto c & & & a &\longmapsto (a, 0). \end{aligned}$$

From the non-graded setting [48, Prop. 2.7],

$$0 \longrightarrow \mathcal{J}(A) \xrightarrow{\theta^{-1} \circ \iota} \mathcal{J}(B) \xrightarrow{\pi \circ \theta} \mathcal{J}(C) \longrightarrow 0$$

is an exact sequence of R -modules, where $\theta : \mathcal{J}(B) \rightarrow \mathcal{J}(A) \oplus \mathcal{J}(C)$ is the R -module homomorphism as above. Then as the maps θ, ι, π are graded R -module homomorphisms, the exact sequence is an exact sequence in $\mathcal{P}\text{gr}(R)$.

Proposition 5.2.5. *Let R be a Γ -graded ring. If R is strongly graded, then $K_0^{\text{gr}}(R) \cong K_0(R_e)$.*

Proof. Since R is strongly graded, we can apply Corollary 5.2.4, which says that the category of graded finitely generated projective modules over R is equivalent to the category of finitely generated projective modules over R_e . By Theorem 2.3.2, each K_i is a functor from the category of exact categories with exact functors to the category of abelian groups. We observed above that \mathcal{J} and \mathcal{G} are exact functors between exact categories, so this implies $K_0(\mathcal{P}\text{gr}(R)) \cong K_0(\mathcal{P}\text{r}(R_e))$ as abelian groups. That is, using the previous notation, $K_0^{\text{gr}}(R) \cong K_0(R_e)$. \square

5.3 Examples

In this section, we consider a specific example of a graded Azumaya algebra A (see Example 5.3.2). We show that $K_0^{\text{gr}}(A) \otimes \mathbb{Z}[1/n]$ is not isomorphic to $K_0^{\text{gr}}(Z(A)) \otimes \mathbb{Z}[1/n]$ for this graded Azumaya algebra A . This leads to the following question, which we will partially answer in Section 5.4 (see Theorem 5.4.4).

Question 5.3.1. Let Γ be an abelian group, R be a commutative Γ -graded ring, and A be a graded Azumaya algebra over its centre R of rank n . When do we have

$$K_0^{\text{gr}}(A) \otimes \mathbb{Z}[1/n] \cong K_0^{\text{gr}}(R) \otimes \mathbb{Z}[1/n]?$$

We now explain the example mentioned above.

Example 5.3.2. Consider the quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$. In Example 1.5.4, we showed that \mathbb{H} is an Azumaya algebra over \mathbb{R} . Then from Example 4.2.3(2), \mathbb{H} is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded division ring, so it is in fact a graded Azumaya algebra, which is strongly $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded.

We can now use Proposition 5.2.5. Here $\mathbb{H}_0 = \mathbb{R}$, so $K_0^{\text{gr}}(\mathbb{H}) \cong K_0(\mathbb{R}) \cong \mathbb{Z}$. The centre $Z(\mathbb{H}) = \mathbb{R}$ is a field and is trivially graded by $\mathbb{Z}_2 \times \mathbb{Z}_2$. By Proposition 5.1.4, $K_0^{\text{gr}}(Z(\mathbb{H})) = K_0^{\text{gr}}(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. We can see that $K_0^{\text{gr}}(\mathbb{H}) \otimes \mathbb{Z}[1/2] \cong \mathbb{Z} \otimes \mathbb{Z}[1/2]$, but $K_0^{\text{gr}}(Z(\mathbb{H})) \otimes \mathbb{Z}[1/2] \cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \otimes \mathbb{Z}[1/2]$, so they are clearly not isomorphic.

We give here another example, which generalises the above example of \mathbb{H} as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded ring.

Example 5.3.3. Let F be a field, ξ be a primitive n -th root of unity and let $a, b \in F^*$. Let

$$A = \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n-1} F x^i y^j$$

be the F -algebra generated by the elements x and y , which are subject to the relations $x^n = a$, $y^n = b$ and $xy = \xi yx$. By [20, Thm. 11.1], A is an n^2 -dimensional central simple algebra over F .

Further, we will show that A forms a graded division ring. Clearly A can be written as a direct sum

$$A = \bigoplus_{(i,j) \in \mathbb{Z}_n \oplus \mathbb{Z}_n} A_{(i,j)}, \quad \text{where } A_{(i,j)} = Fx^i y^j$$

and each $A_{(i,j)}$ is an additive subgroup of A . Using the fact that $\xi^{-kj} x^k y^j = y^j x^k$ for each j, k , with $0 \leq j, k \leq n-1$, we can show that $A_{(i,j)} A_{(k,l)} \subseteq A_{([i+k], [j+l])}$, for $i, j, k, l \in \mathbb{Z}_n$. A non-zero homogeneous element $fx^i y^j \in A_{(i,j)}$ has an inverse

$$f^{-1} a^{-1} b^{-1} \xi^{-ij} x^{n-i} y^{n-j},$$

proving A is a graded division ring. Clearly the support of A is $\mathbb{Z}_n \times \mathbb{Z}_n$, so A is strongly $\mathbb{Z}_n \times \mathbb{Z}_n$ -graded. As for Example 5.3.2, $K_0^{\text{gr}}(A) \cong K_0(A_0) = K_0(F) \cong \mathbb{Z}$. The centre F is trivially graded by $\mathbb{Z}_n \times \mathbb{Z}_n$, so $K_0^{\text{gr}}(Z(A)) \cong \bigoplus_{i=1}^{n^2} \mathbb{Z}_i$ where $\mathbb{Z}_i = \mathbb{Z}$.

Remark 5.3.4. We saw in Example 4.2.3(2) that \mathbb{H} can also be considered as a \mathbb{Z}_2 -graded division ring. So \mathbb{H} is also strongly \mathbb{Z}_2 -graded, and $K_0^{\text{gr}}(\mathbb{H}) \cong K_0(\mathbb{H}_0) = K_0(\mathbb{C}) \cong \mathbb{Z}$. Then $Z(\mathbb{H}) = \mathbb{R}$, which we can consider as a trivially \mathbb{Z}_2 -graded field, so by Proposition 5.1.4, $K_0^{\text{gr}}(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$. We note that for both grade groups, \mathbb{Z}_2 and $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have $K_0^{\text{gr}}(\mathbb{H}) \cong \mathbb{Z}$, but the $K_0^{\text{gr}}(\mathbb{R})$ are different. So the graded K -theory of a graded ring depends not only on the ring, but also on its grade group.

We note that in the above example, the graded K -theory of \mathbb{H} is isomorphic to the usual K -theory of \mathbb{H} . This follows since \mathbb{H} is a division ring, so by Example 2.1.5(1) we have $K_0(\mathbb{H}) \cong \mathbb{Z}$, and we observed above that $K_0^{\text{gr}}(\mathbb{H}) \cong \mathbb{Z}$. But it is not always the case that the graded K -theory and usual K -theory coincide. For \mathbb{R} , we know that $K_0(\mathbb{R}) \cong \mathbb{Z}$. But in the above examples, when \mathbb{R} was trivially graded by \mathbb{Z}_2 (resp. $\mathbb{Z}_2 \times \mathbb{Z}_2$), we had $K_0^{\text{gr}}(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$ (resp. $K_0^{\text{gr}}(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$). Below we will give an example of a graded ring which is not trivially graded, and its graded K -theory is not isomorphic to its usual K -theory.

Example 5.3.5. Let K be a field and let $R = K[x^2, x^{-2}]$. Then R is a \mathbb{Z} -graded field, where R can be written as $R = \bigoplus_{n \in \mathbb{Z}} R_n$, with $R_n = Kx^n$ if n is even and $R_n = 0$ if n is odd. Consider the shifted graded matrix ring $A = M_3(R)(0, 1, 1)$, which has support \mathbb{Z} . Then we will show that A is a graded central simple algebra over R , so by Theorem 4.3.3, A is a graded Azumaya algebra over R .

It is clear that the centre of A is R , and A is finite dimensional over R . We note that if A has a non-zero homogeneous two-sided ideal J , then J is generated by homogeneous elements (see Proposition 4.1.4). Using the elementary matrices, we can show that $J = A$ (see [37, Ex. III.2.9]), so A is graded simple.

Further, we will show that A is a strongly \mathbb{Z} -graded ring. Using Proposition 4.1.3, it is sufficient to show that $I_3 \in A_n A_{-n}$ for all $n \in \mathbb{Z}$. To show this, we note that

$$I_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ x^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & x^{-2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & x^{-2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } I_3 = \begin{pmatrix} 0 & x^{-2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ x^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $I_3 \in A_1 A_{-1}$ and $I_3 \in A_{-1} A_1$, and the required result follows by induction, showing A is strongly graded.

As in the previous examples, using Proposition 5.2.5 we have $K_0^{\text{gr}}(A) \cong K_0(A_0)$. Here $R_0 = K$, so there is a ring isomorphism

$$A_0 = \begin{pmatrix} R_0 & 0 & 0 \\ 0 & R_0 & R_0 \\ 0 & R_0 & R_0 \end{pmatrix} \cong K \times M_2(K).$$

Then

$$K_0^{\text{gr}}(A) \cong K_0(A_0) \cong K_0(K) \oplus K_0(M_2(K)) \cong \mathbb{Z} \oplus \mathbb{Z},$$

since K_0 respects Cartesian products and Morita equivalence. Note that for the usual K -theory of A , $K_0(M_3(R)(0, 1, 1)) \cong K_0(R) = K_0(K[x^2, x^{-2}])$. Using the Fundamental Theorem of Algebraic K -theory [58, Thm. 3.3.3] (see also [48, p. 484]), $K_0(K[x, x^{-1}]) \cong K_0(K) \cong \mathbb{Z}$, and it follows that $K_0(A) \cong \mathbb{Z}$, since $K[x^2, x^{-2}] \cong K[x, x^{-1}]$ as rings. So the K -theory of A is isomorphic to one copy of \mathbb{Z} , which is not the same as the graded K -theory of A .

5.4 Graded \mathcal{D} -functors

Throughout this section, we will assume that Γ is an abelian group, R is a fixed commutative Γ -graded ring and all graded rings, graded modules and graded algebras are also Γ -graded. As mentioned in Remark 4.4.3, in this section we will define multiplication in $\text{End}_R(A)$ to be $g \cdot f = g \circ f$. Let $\mathcal{A}b$ be the category of abelian groups and let $\text{Az}_{\text{gr}}(R)$ denote the category of graded Azumaya algebras graded free over R with graded R -algebra homomorphisms. We recall from Section 4.2 that

$$\Gamma_{M_k(R)}^* = \{(d) \in \Gamma^k : \text{GL}_k(R)[d] \neq \emptyset\},$$

where, for $(d) = (\delta_1, \dots, \delta_k) \in \Gamma^k$, $\text{GL}_k(R)[d]$ consists of invertible $k \times k$ matrices with the ij -entry in $R_{-\delta_i}$ (see page 71).

Definition 5.4.1. An abstract functor $\mathcal{F} : \text{Az}_{\text{gr}}(R) \rightarrow \mathcal{A}b$ is defined to be a *graded \mathcal{D} -functor* if it satisfies the three properties below:

- (1) $\mathcal{F}(R)$ is the trivial group.
- (2) For any graded Azumaya algebra A graded free over R and for any $(d) = (\delta_1, \dots, \delta_k) \in \Gamma_{M_k(R)}^*$, there is a homomorphism

$$\rho : \mathcal{F}(M_k(A)(d)) \longrightarrow \mathcal{F}(A)$$

such that the composition $\mathcal{F}(A) \rightarrow \mathcal{F}(M_k(A)(d)) \rightarrow \mathcal{F}(A)$ is η_k , where $\eta_k(x) =$

x^k .

(3) With ρ as in property (2), then $\ker(\rho)$ is k -torsion.

Note that these properties are well-defined since both R and $M_k(A)(d)$ are graded Azumaya algebras graded free over R . The proof of the theorem below follows in a similar way to that of Theorem 3.1.2.

Theorem 5.4.2. *Let A be a graded Azumaya algebra which is graded free over its centre R of dimension n , such that A has a homogeneous basis with degrees $(\delta_1, \dots, \delta_n)$ in $\Gamma_{M_n(R)}^*$. Then $\mathcal{F}(A)$ is n^2 -torsion, where \mathcal{F} is a graded \mathcal{D} -functor.*

Proof. Let $\{a_1, \dots, a_n\}$ be a homogeneous basis for A over R , and let $(d) = (\deg(a_1), \dots, \deg(a_n)) \in \Gamma_{M_n(R)}^*$. Since R is a graded Azumaya algebra over itself, by (2) in the definition of a graded \mathcal{D} -functor, there is a homomorphism $\rho : \mathcal{F}(M_n(R)(d)) \rightarrow \mathcal{F}(R)$. But $\mathcal{F}(R)$ is trivial by property (1) and therefore the kernel of ρ is $\mathcal{F}(M_n(R)(d))$ which is, by (3), n -torsion. Further, the graded R -algebra isomorphism $A \otimes_R A^{\text{op}} \cong_{\text{gr}} \text{End}_R(A)$ from the definition of a graded Azumaya algebra, combined with the graded isomorphism $\text{End}_R(A) \cong_{\text{gr}} M_n(R)(d)$, induces an isomorphism $\mathcal{F}(A \otimes_R A^{\text{op}}) \cong \mathcal{F}(M_n(R)(d))$. So $\mathcal{F}(A \otimes_R A^{\text{op}})$ is also n -torsion.

In the category $\text{Az}_{\text{gr}}(R)$, the two graded R -algebra homomorphisms $i : A \rightarrow A \otimes_R A^{\text{op}}$ and $r : A^{\text{op}} \rightarrow \text{End}_R(A^{\text{op}}) \rightarrow M_n(R)(d)$ induce group homomorphisms $i : \mathcal{F}(A) \rightarrow \mathcal{F}(A \otimes_R A^{\text{op}})$ and $r : \mathcal{F}(A \otimes_R A^{\text{op}}) \rightarrow \mathcal{F}(A \otimes_R M_n(R)(d))$, where $\mathcal{F}(A \otimes_R M_n(R)(d)) \cong \mathcal{F}(M_n(A)(d))$. Consider the following diagram

$$\begin{array}{ccc}
 \mathcal{F}(A) & & \\
 \downarrow i & \searrow \eta_n & \\
 \mathcal{F}(A \otimes_R A^{\text{op}}) & & \\
 \downarrow r & & \searrow \rho \\
 \mathcal{F}(M_n(A)(d)) & \xrightarrow{\quad} & \mathcal{F}(A)
 \end{array}$$

which is commutative by property (2). It follows that $\mathcal{F}(A)$ is n^2 -torsion. \square

For a graded ring A , let $\mathcal{Pgr}(A)$ be the category of graded finitely generated projective A -modules. Then as for K_0^{gr} in Section 5.1, we define

$$K_i^{\text{gr}}(A) = K_i(\mathcal{Pgr}(A))$$

for $i \geq 0$, where $K_i(\mathcal{Pgr}(A))$ is the Quillen K -group of the exact category $\mathcal{Pgr}(A)$ (see Section 2.3).

Let A be a graded ring with graded centre R . Then the graded ring homomorphism $R \rightarrow A$ induces an exact functor $A \otimes_R - : \mathcal{Pgr}(R) \rightarrow \mathcal{Pgr}(A)$, which in turn induces a group homomorphism $K_i^{\text{gr}}(R) \rightarrow K_i^{\text{gr}}(A)$. Then we have an exact sequence

$$1 \longrightarrow \text{ZK}_i^{\text{gr}}(A) \longrightarrow K_i^{\text{gr}}(R) \longrightarrow K_i^{\text{gr}}(A) \longrightarrow \text{CK}_i^{\text{gr}}(A) \longrightarrow 1 \quad (5.1)$$

where $\text{ZK}_i^{\text{gr}}(A)$ and $\text{CK}_i^{\text{gr}}(A)$ are the kernel and cokernel of the map $K_i^{\text{gr}}(R) \rightarrow K_i^{\text{gr}}(A)$ respectively. We will show that CK_i^{gr} can be regarded as the following functor

$$\begin{aligned} \text{CK}_i^{\text{gr}} : \text{Az}_{\text{gr}}(R) &\longrightarrow \mathcal{A}b \\ A &\longmapsto \text{CK}_i^{\text{gr}}(A). \end{aligned}$$

For a graded Azumaya algebra A graded free over R , clearly $\text{CK}_i^{\text{gr}}(A) = \text{coker}(K_i^{\text{gr}}(R) \rightarrow K_i^{\text{gr}}(A))$ is an abelian group. Consider graded Azumaya algebras A, A' graded free over R and a graded R -algebra homomorphism $f : A \rightarrow A'$. Then there is an induced exact functor $A' \otimes_A - : \mathcal{Pgr}(A) \rightarrow \mathcal{Pgr}(A')$, which induces a group homomorphism $f_* : K_i^{\text{gr}}(A) \rightarrow K_i^{\text{gr}}(A')$. We have an exact functor

$$A' \otimes_A (A \otimes_R -) : \mathcal{Pgr}(R) \longrightarrow \mathcal{Pgr}(A) \longrightarrow \mathcal{Pgr}(A').$$

As the map f restricted to R is the identity map, the induced functor $\mathcal{Pgr}(R) \rightarrow \mathcal{Pgr}(R)$ is also the identity. So it induces the identity map $K_i^{\text{gr}}(R) \rightarrow K_i^{\text{gr}}(R)$. We

have an exact functor

$$A' \otimes_R - : \mathcal{P}\text{gr}(R) \longrightarrow \mathcal{P}\text{gr}(R) \longrightarrow \mathcal{P}\text{gr}(A').$$

Since these two functors from $\mathcal{P}\text{gr}(R)$ to $\mathcal{P}\text{gr}(A')$ are isomorphic, they induce the same map on the level of the K -groups by Theorem 2.3.2.

Since K_i is a functor from the category of exact categories to the abelian groups, the following diagram is commutative

$$\begin{array}{ccccc} K_i^{\text{gr}}(R) & \longrightarrow & K_i^{\text{gr}}(A) & \longrightarrow & \text{CK}_i^{\text{gr}}(A) \\ \text{id} \downarrow & & \downarrow f_* & & \downarrow \text{CK}_i^{\text{gr}}(f) \\ K_i^{\text{gr}}(R) & \longrightarrow & K_i^{\text{gr}}(A') & \longrightarrow & \text{CK}_i^{\text{gr}}(A') \end{array}$$

Then it is routine to check that CK_i^{gr} forms the required functor. Similarly ZK_i^{gr} can be regarded as the following functor

$$\begin{aligned} \text{ZK}_i^{\text{gr}} : \text{Az}_{\text{gr}}(R) &\longrightarrow \mathcal{A}b \\ A &\longmapsto \text{ZK}_i^{\text{gr}}(A). \end{aligned}$$

Proposition 5.4.3. *With CK_i^{gr} defined as above, CK_i^{gr} is a graded \mathcal{D} -functor.*

Proof. Property (1) is clear, since R is commutative so $K_i^{\text{gr}}(Z(R)) \rightarrow K_i^{\text{gr}}(R)$ is the identity map. For property (2), let A be a graded Azumaya algebra graded free over R and let $(d) = (\delta_1, \dots, \delta_k) \in \Gamma_{M_k(R)}^*$. Then there are functors:

$$\begin{aligned} \phi : \mathcal{P}\text{gr}(A) &\longrightarrow \mathcal{P}\text{gr}(M_k(A)(d)) \\ X &\longmapsto M_k(A)(d) \otimes_A X \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \psi : \mathcal{P}\text{gr}(M_k(A)(d)) &\longrightarrow \mathcal{P}\text{gr}(A) \\ Y &\longmapsto A^k(-d) \otimes_{M_k(A)(d)} Y. \end{aligned} \quad (5.3)$$

The functor ϕ induces a homomorphism from $K_i^{\text{gr}}(A)$ to $K_i^{\text{gr}}(M_k(A)(d))$. By the graded version of the Morita Theorems (see Proposition 4.5.4), the functor ψ establishes a natural equivalence of categories, so it induces an isomorphism from $K_i^{\text{gr}}(M_k(A)(d))$ to $K_i^{\text{gr}}(A)$.

For $X \in \mathcal{P}\text{gr}(A)$, $\psi \circ \phi(X) \cong_{\text{gr}} X^k(-d)$. We will use a similar argument to that of Proposition 4.2.2 to show that $X^k(-d) \cong_{\text{gr}} X^k$. Since $(d) \in \Gamma_{M_k(R)}^*$, there is $r = (r_{ij}) \in \text{GL}_k(R)[d]$. Then there is a graded A -module homomorphism

$$\begin{aligned} \mathcal{L}_r : X^k &\longrightarrow X^k(-d) \\ (x_1, \dots, x_k) &\longmapsto r(x_1, \dots, x_k). \end{aligned}$$

Since r is invertible there is an inverse matrix $t \in \text{GL}_k(R)$, and there is an A -module homomorphism $\mathcal{L}_t : X^k(-d) \rightarrow X^k$ which is an inverse of \mathcal{L}_r . So \mathcal{L}_r is a graded A -module isomorphism. By the remarks after Theorem 2.3.5, K_i are functors which respect direct sums, so this induces a multiplication by k on the level of the K -groups.

The exact functors (5.2) and (5.3) induce the following commutative diagram:

$$\begin{array}{ccccccc} K_i^{\text{gr}}(R) & \longrightarrow & K_i^{\text{gr}}(A) & \longrightarrow & \text{CK}_i^{\text{gr}}(A) & \longrightarrow & 1 \\ \downarrow = & & \downarrow \phi & & \downarrow & & \\ K_i^{\text{gr}}(R) & \longrightarrow & K_i^{\text{gr}}(M_k(A)(d)) & \longrightarrow & \text{CK}_i^{\text{gr}}(M_k(A)(d)) & \longrightarrow & 1 \\ \downarrow \eta_k & & \downarrow \cong \psi & & \downarrow \rho & & \\ K_i^{\text{gr}}(R) & \longrightarrow & K_i^{\text{gr}}(A) & \longrightarrow & \text{CK}_i^{\text{gr}}(A) & \longrightarrow & 1 \end{array}$$

where composition of the columns are η_k , proving property (2). A diagram chase verifies that property (3) also holds. \square

A similar proof shows that ZK_i^{gr} is also a graded \mathcal{D} -functor. We now show that for a graded Azumaya algebra which is graded free over its centre, its graded K -theory is essentially the same as the graded K -theory of its centre.

Theorem 5.4.4. *Let A be a graded Azumaya algebra which is graded free over its centre R of dimension n , such that A has a homogeneous basis with degrees $(\delta_1, \dots, \delta_n)$ in $\Gamma_{M_n(R)}^*$. Then for any $i \geq 0$,*

$$K_i^{\text{gr}}(A) \otimes \mathbb{Z}[1/n] \cong K_i^{\text{gr}}(R) \otimes \mathbb{Z}[1/n].$$

Proof. Proposition 5.4.3 shows that CK_i^{gr} (and in the same manner ZK_i^{gr}) is a graded \mathcal{D} -functor, and thus by Theorem 5.4.2 $CK_i^{\text{gr}}(A)$ and $ZK_i^{\text{gr}}(A)$ are n^2 -torsion abelian groups. Tensoring the exact sequence (5.1) by $\mathbb{Z}[1/n]$, since $CK_i^{\text{gr}}(A) \otimes \mathbb{Z}[1/n]$ and $ZK_i^{\text{gr}}(A) \otimes \mathbb{Z}[1/n]$ vanish, the result follows. \square

It remains as a question when this result holds for a graded Azumaya algebra of constant rank.

Question 5.4.5. Let Γ be an abelian group, R be a commutative Γ -graded ring, and A be a graded Azumaya algebra over its centre R of rank n . When is it true that for any $i \geq 0$,

$$K_i^{\text{gr}}(A) \otimes \mathbb{Z}[1/n] \cong K_i^{\text{gr}}(R) \otimes \mathbb{Z}[1/n]?$$

Remark 5.4.6. Using Example 5.3.2, we remark here that the graded Azumaya algebra \mathbb{H} does not satisfy the conditions of Theorem 5.4.4. Suppose \mathbb{H} does satisfy these conditions; that is, suppose there is a homogeneous basis $\{a_1, \dots, a_4\}$ for \mathbb{H} over \mathbb{R} , such that the elements of the basis have degrees $(\delta_1, \dots, \delta_4)$ in $\Gamma_{M_4(\mathbb{R})}^*$. So there exists a matrix $r \in \text{GL}_4(\mathbb{R})[d]$. Since $\text{Supp}(\mathbb{R}) = 0$, then as each row of r must contain a non-zero element, this implies $\delta_i = 0$ for each i . But this would imply that the support of \mathbb{H} is also 0, which clearly is a contradiction. So such a homogeneous basis for \mathbb{H} does not exist.

Corollary 5.4.7. *Let A be an Azumaya algebra free over its centre R of dimension n . Then for any $i \geq 0$,*

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. By taking Γ to be the trivial group, this follows immediately from Theorem 5.4.4. □

Chapter 6

Additive Commutators

In 1905 Wedderburn proved that a finite division ring is a field. This is an example of a commutativity theorem; that is, it is a theorem which states certain conditions under which a given ring is commutative. Since then, Wedderburn's result has motivated many, more general commutativity theorems (see [42, §13]). Both additive and multiplicative commutators play an important role in these theorems. In this chapter we consider additive commutators in the setting of graded division algebras.

We begin Section 6.1 with two results involving the support of a graded division ring. We then show that some commutativity theorems involving additive commutators hold in the graded setting. We give a counter-example to show that one such commutativity theorem for multiplicative commutators does not hold.

In Section 6.2, we show that in the setting of graded division algebras, the reduced trace exists and it is a graded map. In Section 6.3, we recall some results from the non-graded setting which will be used in the final section. We end the chapter by considering, in Section 6.4, the quotient division ring QD of a graded division ring D . We show how the submodule generated by the additive commutators in QD relates to that of D (see Corollary 6.4.5).

6.1 Homogeneous additive commutators

Throughout this chapter, let Γ be an abelian group unless otherwise stated. We recall from Section 4.1 that the support of a graded ring $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ is defined to be the set

$$\text{Supp}(R) = \Gamma_R = \{\gamma \in \Gamma : R_\gamma \neq \{0\}\}.$$

It follows that a graded ring R is zero if and only if $\text{Supp}(R) = \emptyset$.

Let D be a Γ -graded division ring with centre F . Since Γ is an abelian group, the centre of D is a graded subring of D (see Section 4.1). A *homogeneous additive commutator* of D is defined to be an element of the form $ab - ba$ where $a, b \in D^h$. Throughout this chapter, we will use the notation $[a, b] = ab - ba$ and $[D, D]$ is the graded submodule of D generated by all homogeneous additive commutators of D .

We note that in this chapter we will consider the group Γ to be an abelian group, unless stated otherwise. This is the natural setting to consider homogeneous additive commutators. If Γ is not abelian, then a given homogeneous additive commutator may not be a homogeneous element in D .

A *graded division algebra* D is defined to be a graded division ring with centre F such that $[D : F] < \infty$. Note that since F is a graded field, D has a finite homogeneous basis over F . A graded division algebra D over its centre F is said to be *unramified* if $\Gamma_D = \Gamma_F$ and *totally ramified* if $D_0 = F_0$. The following lemma considers the support of $[D, D]$; the proof of part (2) is due to Hazrat.

Lemma 6.1.1. *Let $D = \bigoplus_{\gamma \in \Gamma} D_\gamma$ be a graded division algebra over its centre F .*

1. *If D is totally ramified, then $\emptyset \neq \text{Supp}([D, D]) \subsetneq \Gamma_D$.*
2. *If D is not totally ramified, then $\text{Supp}(D) = \text{Supp}([D, D])$.*

Proof. (1): Clearly $\emptyset \neq \text{Supp}([D, D]) \subseteq \Gamma_D$. Since $D_0 = F_0 = Z(D) \cap D_0$ we have $D_0 \subseteq Z(D)$. Suppose $0 \in \text{Supp}([D, D])$. Then there is an element $\sum_i (x_i y_i - y_i x_i) \in [D, D]$, with $\deg(x_i) + \deg(y_i) = 0$ for all i . If $x_i y_i - y_i x_i = 0$ for all i , then clearly

the sum is also zero. Thus there are non-zero homogeneous elements $x \in D_\gamma$, $y \in D_\delta$ with $0 \neq xy - yx \in D_0$ and $\gamma + \delta = 0$.

Then $(xy - yx)y^{-1} \neq 0$, as $y^{-1} \in D_{-\delta} \setminus 0$ and $xy - yx \in D_0 \setminus 0$, so their product is a non-zero homogeneous element of degree $-\delta$. Since

$$(xy - yx)y^{-1} = xyy^{-1} - yxy^{-1} = y^{-1}yx - yxy^{-1},$$

we have $y^{-1}(yx) \neq (yx)y^{-1}$; that is $yx \notin Z(D)$. Since $yx \in D_0$, this contradicts the fact that $D_0 = F_0$, so $0 \notin \text{Supp}([D, D])$.

(2): It is clear that $\text{Supp}([D, D]) \subseteq \Gamma_D$. For the reverse containment, for $\gamma \in \Gamma_D$ we will show that there is an $x \in D_\gamma$ which does not commute with some $y \in D_\delta$ for some $\delta \in \Gamma_D$. Suppose not, then $D_\gamma \subseteq Z(D)$, so $D_\gamma = F_\gamma$. Let $x \in D_\gamma$, $d \in D_0$, $y \in D_\delta$ be arbitrary non-zero elements. Then

$$x(dy) = (dy)x = d(yx) = d(xy) = (dx)y = y(dx) = (yd)x = x(yd).$$

So for all $d \in D_0$, $y \in D_\delta$ we have $x(dy) = x(yd)$. Since x is a non-zero homogeneous element, it is invertible. This implies $dy = yd$, so $D_0 = F_0$ contradicting the fact that D is not totally ramified. Then there is an $x \in D_\gamma$ which does not commute with $y \in D_\delta$, so $xyy^{-1} - y^{-1}xy \neq 0$ proving $\gamma \in \text{Supp}([D, D])$. \square

Example 6.1.2. Let \mathbb{H} be the real quaternion algebra. We saw in Example 4.2.3(2) that \mathbb{H} forms a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded division ring, where its centre is \mathbb{R} . Then $\text{Supp}(\mathbb{H}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\text{Supp}(\mathbb{R}) = (0, 0)$. We can show that $\text{Supp}([\mathbb{H}, \mathbb{H}]) = \{(1, 0), (0, 1), (1, 1)\}$.

Let D be a graded division ring and let F be a graded subfield of D which is contained in the centre of D . We know that $F_0 = F \cap D_0$ is a field and D_0 is a division ring. The group of invertible homogeneous elements of D , denoted by D^{h*} , is equal to $D^h \setminus 0$. Considering D as a graded F -module, since F is a graded field, there is a uniquely defined dimension $[D : F]$ by Theorem 4.2.4. Note that $\Gamma_F \subseteq \Gamma_D$, so Γ_F is a normal subgroup of Γ_D .

The proposition below has been proven by Hwang, Wadsworth [38, Prop. 2.2] for two graded fields $R \subseteq S$ with a torsion-free abelian grade group.

Proposition 6.1.3. *Let D be a graded division ring and let F be a graded subfield of D which is contained in the centre of D . Then*

$$[D : F] = [D_0 : F_0] |\Gamma_D : \Gamma_F|.$$

Proof. Let $\{x_i\}_{i \in I}$ be a basis for D_0 over F_0 . Consider the cosets of Γ_D over Γ_F and take a transversal $\{\delta_j\}_{j \in J}$ for these cosets, where $\delta_j \in \Gamma_D$. Take $\{y_j\}_{j \in J} \subseteq D^{h*}$ such that $\deg(y_j) = \delta_j$ for each j . We will show that $\{x_i y_j\}$ is a basis for D over F .

Consider the map

$$\begin{aligned} \psi : D^{h*} &\longrightarrow \Gamma_D / \Gamma_F \\ d &\longmapsto \deg(d) + \Gamma_F. \end{aligned}$$

This is a group homomorphism with kernel $D_0 F^{h*}$, since for any $d \in \ker(\psi)$ there is some $f \in F^{h*}$ with $df^{-1} \in D_0$. Let $d \in D$ be arbitrary. Then $d = \sum_{\gamma \in \Gamma} d_\gamma$ where $d_\gamma \in D_\gamma$ and $\psi(d_\gamma) = \gamma + \Gamma_F = \delta_j + \Gamma_F$ for some δ_j in the transversal of Γ_D over Γ_F . Then there is some y_j with $\deg(y_j) = \delta_j$ and $d_\gamma y_j^{-1} \in \ker(\psi)$. So $d_\gamma y_j^{-1} = \sum_k a_k g_k$ for $g_k \in F^{h*}$ and $a_k = \sum_i r_i^{(k)} x_i$ with $r_i^{(k)} \in F_0$. It follows that d can be written as an F -linear combination of the elements of $\{x_i y_j\}$.

To show linear independence, suppose $\sum_{i=1}^n r_i x_i y_i = 0$ for $r_i \in F$. Since the homogeneous components of D are disjoint, we can take a homogeneous component of this sum, say $\sum_{k=1}^m r_k x_k y_k$ where $\deg(r_k x_k y_k) = \alpha$. Then $\deg(r_k) + \deg(y_k) = \alpha$ for all k , so all of the y_k are the same. This implies that $\sum_k r_k x_k = 0$, where all of the r_k have the same degree. If $r_k = 0$ for all k then we are done. Otherwise, for some $r_l \neq 0$, we have $\sum_k (r_l^{-1} r_k) x_k = 0$. Since $\{x_i\}$ forms a basis for D_0 over F_0 , this implies $r_k = 0$ for all k . \square

We include below a number of results involving homogeneous additive commu-

tators. The proofs follow in exactly the same way as the proofs of the equivalent non-graded results (see [43, §13]).

Lemma 6.1.4. *Let D be a graded division ring. If all of the homogeneous additive commutators of D are central, then D is a graded field.*

Proof. Let $y \in D^h$ be an arbitrary homogeneous element of D . Then by assumption, y commutes with all homogeneous additive commutators, and it follows that y commutes with all (non-homogeneous) additive commutators of D . We will show $y \in Z(D)$.

Suppose $y \notin Z(D)$. Then there exists $x \in D^h$ such that $[x, y] \neq 0$. We have $[x, xy] = x[x, y]$ with $[x, xy]$ and $[x, y]$ non-zero. Since y commutes with $[x, xy]$ and $[x, y]$, it follows that $(yx - xy)[x, y] = 0$. As we assumed $[x, y] \neq 0$, it is a non-zero homogeneous element of D , so is invertible. So $yx - xy = 0$; that is, $[x, y] = 0$ contradicting our choice of x . It follows that D is a commutative graded division ring, as required. \square

We include an alternative proof to Lemma 6.1.4 in Section 6.4. This alternative proof uses the relation between a graded division ring and its quotient division ring combined with the non-graded result.

Theorem 6.1.5. *Let D be a graded division ring with centre F . Then the smallest graded division subring over F generated by homogeneous additive commutators is D .*

Proof. Let $E = \bigoplus_{\gamma \in \Gamma} E_\gamma$ be the smallest graded F -division subring containing all homogeneous additive commutators. Then clearly we have $F \subseteq E \subseteq D$. To show $D = E$, it is sufficient to show that $D^h \subseteq E^h$. Let $x \in D^h \setminus F^h$. Then there exists $y \in D^h$ such that $xy \neq yx$. We have $[x, xy] = x[x, y]$ with $[x, xy], [x, y] \in E^h$, since they are non-zero homogeneous additive commutators. Then $x = [x, xy] \cdot [x, y]^{-1} \in E^h$, as required. \square

Proposition 6.1.6. *Let $K \subseteq D$ be graded division rings, with $[D, K] \subseteq K$. If $\text{char } K \neq 2$, then $K \subseteq Z(D)$.*

Proof. First note that the condition $[D, K] \subseteq K$ is equivalent to $[D^h, K^h] \subseteq K^h$. Let $a \in D^h \setminus K^h$ and $c \in K^h$. We will show $ac = ca$. We have $[a, [a, c]] + [a^2, c] = 2a[a, c] \in K$. If $[a, c] \neq 0$, then, since $\text{char } K \neq 2$, this implies $a \in K$, contradicting our choice of a . Thus $[a, c] = 0$.

Now let $b, c \in K^h$. Consider $a \in D^h \setminus K^h$. Then $a, ab \in D^h \setminus K^h$ and we have shown that $[a, c], [ab, c] = 0$. Then $[b, c] = a^{-1} \cdot [ab, c] = 0$. It follows that $K \subseteq Z(D)$. \square

The above results show that, in some aspects, the behaviour of homogeneous additive commutators in graded division rings is similar to that of additive commutators in division rings. However, this analogy seems to fail for multiplicative commutators of graded division rings. For example, in the setting of division rings the Cartan-Brauer-Hua theorem is the multiplicative version of Proposition 6.1.6 and its proof involves multiplicative commutators. In the non-graded setting, the Cartan-Brauer-Hua theorem states:

Let D and K be division rings, with $K \subseteq D$. Suppose that K^* is a normal subgroup of D^* and $K \neq D$. Then $K \subseteq Z(D)$.

This theorem does not hold in the setting of graded division rings, as is shown by the following counterexample.

Example 6.1.7. Let D be a division ring and let $R = D[x, x^{-1}]$. Then we saw in Example 4.2.3(1) that R is a graded division ring. Let $K = R_0$. Then K^* is a normal subgroup of R^* and $K \neq R$. Choose $a, b \in D$ with $ab \neq ba$. Then $ax^0 \in K$ and $bx^n \in R$, and we have $(ax^0)(bx^n) = abx^n \neq bax^n = (bx^n)(ax^0)$. So $ax^0 \notin Z(R)$, proving that $K \not\subseteq Z(R)$.

Consider a homogeneous multiplicative commutator $xyx^{-1}y^{-1}$ in a graded division ring D . Since $x, y \in D^h$ and $\deg(x^{-1}) = -\deg(x)$, we note that $\deg(xyx^{-1}y^{-1}) =$

0; that is, $xyx^{-1}y^{-1}$ is in the zero-homogeneous component of D . This suggests that there are “too few” multiplicative commutators to affect the structure of the division ring.

6.2 Graded splitting fields

For a graded division algebra D over its centre F , we will show that D is split by any graded maximal subfield L of D . This allows us to construct the reduced characteristic polynomial of D over F .

Lemma 6.2.1 (Graded Schur’s Lemma). *Let R be a graded ring and let M be a graded R -module. If M is graded simple, then $\text{END}_R(M)$ is a graded division ring.*

Proof. We know from page 76 that $\text{END}_R(M)$ is a graded ring. If f is a nonzero homogeneous endomorphism, then $\ker(f) \neq M$ and $\text{im}(f) \neq 0$. Since $\ker(f)$ and $\text{im}(f)$ are both graded submodules of M , which is graded simple, it follows that $\ker(f) = 0$ and $\text{im}(f) = M$. Thus f is a graded isomorphism, and hence is invertible. \square

In the following theorem, we rewrite Rieffel’s proof of Wedderburn’s Theorem in the graded setting (see [39, Prop. 1.3(a)]). See [51, Thm. 2.10.10] for a more general version of the theorem.

Theorem 6.2.2. *Let A be a graded central simple algebra over a graded field R . Then there is a graded division algebra E over R such that $A \cong_{\text{gr}} M_n(E)(d)$ for some $(d) = (\delta_1, \dots, \delta_n) \in \Gamma^n$.*

Proof. Take a minimal nonzero homogeneous right ideal L of A (which exists since $[A : R] < \infty$). Then L is a graded simple right A -module, as it has no nonzero proper graded right A -submodules. Let $E = \text{End}_A(L)$, where $\text{End}_A(L) = \text{END}_A(L)$ since L is finitely generated as a graded A -module. Then E is a graded division ring, by the graded Schur’s Lemma, and $[E : R] \leq [\text{End}_R(L) : R] < \infty$. Then L is

a graded free left E -module, with a homogeneous base, say b_1, \dots, b_n , of L over E . Then by Theorem 4.4.2, $\text{End}_E(L) \cong_{\text{gr}} M_n(E)(d)$, where $(d) = (\deg(b_1), \dots, \deg(b_n))$. Consider the map

$$\begin{aligned} i : A &\longrightarrow \text{End}_E(L) \\ a &\longmapsto i(a) : L \rightarrow L \\ x &\mapsto xa \end{aligned}$$

We will show that i is a graded R -algebra isomorphism. Firstly note that $i(a)$ is an element of $\text{End}_E(L)$, since for all $e \in E$, $x \in L$ we have

$$(i(a))(ex) = (ex)a = (e(x))a = e(xa) = e((i(a))(x)).$$

It can be easily shown that i is a graded R -algebra homomorphism. Since A is graded simple, it follows that i is injective. Note that $i(A)$ contains the identity element of $\text{End}_E(L)$. Then to prove surjectivity, it suffices to show that $i(A)$ is a homogeneous right ideal of $\text{End}_E(L)$.

For $y \in L$, let $\mathcal{L}_y : L \rightarrow L; x \mapsto yx$ so that $\mathcal{L}_y \in \text{End}_A(L)$. Then for $f \in \text{End}_E(L)^h$, $x \in L$ we have

$$f(yx) = f(\mathcal{L}_y(x)) = \mathcal{L}_y(f(x)) = y(f(x)).$$

It follows that $(i(x) \cdot f)(y) = (f \circ i(x))(y) = (i(f(x)))(y)$, which implies that $i(L)$ is a right ideal of $\text{End}_E(L)$. Also, $i(L) = \bigoplus_{\gamma \in \Gamma} (i(L) \cap (\text{End}_E(L))_\gamma)$ giving that the ideal is homogeneous. The two-sided homogeneous ideal of A generated by L is $ALA = AL$, and since A is graded simple, $AL = A$. Thus $i(A) = i(AL) = i(A)i(L)$ is a homogeneous right ideal of $\text{End}_E(L)$, as required. \square

Let D be a graded division algebra over its centre F and let L be any graded subfield of D containing F . We define a grading on $L[x]$ as follows. Let $\theta \in \Gamma_D$ and

let

$$L[x]^\theta = \bigoplus_{\gamma \in \Gamma} L[x]_\gamma, \text{ where } L[x]_\gamma = \left\{ \sum a_i x^i : a_i \in L^h, \deg(a_i) + i\theta = \gamma \right\}.$$

Then $L[x]^\theta$ forms a graded ring, and $x \in L[x]^\theta$ is homogeneous of degree θ . Let $Z_D(L) = \{d \in D : dl = ld \text{ for all } l \in L\}$ denote the centraliser of L in D . Then $Z_D(L)$ is a graded subring of D and it is a graded L -algebra. For any $c \in Z_D(L)^h$ of degree θ , let

$$L[c] = \{f(c) : f(x) \in L[x]^\theta\}.$$

Then $L[c]$ forms a graded ring with $L \subseteq L[c]$, and we note that $L[c]$ is commutative since $c \in Z_D(L)^h$. The map $L[x]^\theta \rightarrow L[c]; f(x) \mapsto f(c)$ is a graded ring homomorphism.

Further we will show that $L[c]$ is in fact a graded field. Let $a \in L[c]_\gamma$ be a non-zero element, and consider the γ -shifted L -module $L[c](\gamma)$. The map $\mathcal{L}_a : L[c] \rightarrow L[c](\gamma); l \mapsto al$ is a graded L -module homomorphism, which is injective since a is invertible in D . Then $\dim_L(\text{im}(\mathcal{L}_a)) = \dim_L(L[c])$. We will show that $\dim_L(L[c]) < \infty$.

Since $c \in D$ and $[D : F] < \infty$, we have that c is algebraic over F , and thus is algebraic over L . So it has a minimal polynomial $h(x) = l_0 + l_1x + \cdots + l_kx^k$, and the set $\{1, c, c^2, \dots, c^{k-1}\}$ generates $L[c]$ over L . If they are not all linearly independent, we can write 1 as an L -linear combination of the others and remove it from the set, leaving a set which still generates $L[c]$ over L . Repeating this process will give a linearly independent spanning set for $L[c]$ over L .

So $\dim_L(L[c]) < \infty$, and \mathcal{L}_a is surjective by dimension count. Then there is a graded L -module homomorphism ψ which is the inverse of \mathcal{L}_a , and $\psi(1_L)$ is the inverse of a . Since $L[c]$ is also commutative, it is a graded field.

The following result is in [20, p. 40] in the non-graded setting.

Theorem 6.2.3. *Let D be a graded division algebra over a graded field F , and let L be a graded subfield of D . Then L is a graded maximal subfield if and only if*

$$Z_D(L) = L.$$

Proof. If $Z_D(L) = L$, then for any graded subfield L' of D containing L we have $L' \subseteq Z_D(L)$. So L is a graded maximal subfield. Conversely, assume L is graded maximal. Then for any homogeneous $c \in Z_D(L)$, $L[c]$ forms a graded field. By the maximality of L , we must have $L[c] = L$ and so $c \in L$. Then it follows that $Z_D(L) = L$. \square

Corollary 6.2.4. *Let D be a graded division algebra with centre F and let L be a graded maximal subfield of D . Then*

$$D \otimes_F L \cong_{\text{gr}} M_n(L)(d)$$

for some $n \in \mathbb{N}$ and some $d = (\delta_1, \dots, \delta_n) \in \Gamma^n$.

Proof. As graded F -modules, we have $D \otimes_F L \cong_{\text{gr}} L \otimes_F D$. Since D is a graded division algebra over the graded field F , and L is graded simple, Theorems 4.3.1, 4.3.2 give that $L \otimes_F D$ is graded central simple over $Z(L) = L$. We have that D is a graded simple right module over $L \otimes_F D$, with the right action $x(l \otimes d) = lxd$ for $d, x \in D$, $l \in L$. Then by Theorem 6.2.2, $E := \text{End}_{L \otimes_F D}(D)$ is a graded division algebra over L , such that $L \otimes_F D \cong_{\text{gr}} M_n(E)(d)$ for some $d = (\delta_1, \dots, \delta_n) \in \Gamma^n$. From Theorem 6.2.3, $Z_D(L) = L$, so it remains to show $Z_D(L) \cong_{\text{gr}} E$. Define

$$\begin{aligned} \psi : Z_D(L) &\rightarrow \text{End}_{L \otimes_F D}(D) \\ d &\mapsto \psi(d) : D \rightarrow D \\ x &\mapsto dx \end{aligned}$$

It can be easily shown that ψ is a graded L -algebra homomorphism, which is injective since L is graded simple as a graded ring. Let $f \in (\text{End}_{L \otimes_F D}(D))_\gamma$ be a homogeneous map. Then $f(1) \in D_\gamma$ and since $f(1)\ell = \ell f(1)$ for all $\ell \in L^h$, we have $f(1) \in Z_D(L)$. For $x \in D$,

$$f(x) = f(1 \cdot (1 \otimes x)) = f(1)(1 \otimes x) = f(1)x = (\psi(f(1)))(x).$$

So $f = \psi(f(1))$, proving that ψ is surjective. It follows that $M_n(E)(d) \cong_{\text{gr}} M_n(L)(d)$, completing the proof. \square

Suppose D is a graded division algebra over a graded field F . We will show that there exists a graded maximal subfield of D . For any $a \in D^h \setminus F^h$, we have shown above that $F[a]$ is a graded field, with $F \subseteq F[a] \subseteq D$. Consider

$$X = \{L : L \text{ is a graded subfield of } D \text{ with } F \subsetneq L\}.$$

This is a non-empty set since $F[a] \in X$ and it is partially ordered with inclusion. Every chain $L_1 \subseteq L_2 \subseteq \dots$ in X has an upper bound $\bigcup L_i \in X$. By Zorn's Lemma, X has a maximal element, so there is a graded maximal subfield of D .

Then Corollary 6.2.4 shows that a graded maximal subfield L of D splits D ; that is, $j : D \otimes L \cong_{\text{gr}} M_n(L)(d)$. As in the non-graded setting, for an element $d \in D$ we define the reduced characteristic polynomial as

$$\begin{aligned} \text{char}_{D/F}(d, x) &= \det(xI_n - j(d \otimes 1)) \\ &= x^n - \text{Trd}_D(d)x^{n-1} + \dots + (-1)^n \text{Nrd}_D(d), \end{aligned}$$

where $\text{Trd}_D(d) = \text{trace}(j(d \otimes 1))$ is the reduced trace of d and $\text{Nrd}_D(d) = \det(j(d \otimes 1))$ is the reduced norm.

Since L is a graded module over a graded field F , by Proposition 4.2.4, it is graded free and therefore free over F . It follows that L is faithfully flat over F , where F is a ring and D and L are F -algebras, so we can apply [41, Lemma III.1.2.1]. This shows that the reduced characteristic polynomial lies in $F[x]$ and it is independent of the choice of j and L . We note that the reduced norm and reduced trace satisfy the following properties.

Corollary 6.2.5. *Reduced norm and trace satisfy the following rules:*

$$\begin{aligned} \text{Nrd}_D(ab) &= \text{Nrd}_D(a)\text{Nrd}_D(b) & \text{Trd}_D(a+b) &= \text{Trd}_D(a) + \text{Trd}_D(b) \\ \text{Nrd}_D(ra) &= r^n \text{Nrd}_D(a) & \text{Trd}_D(ra) &= r \text{Trd}_D(a) \\ & & \text{Trd}_D(ab) &= \text{Trd}_D(ba) \end{aligned}$$

for all $a, b \in D$ and $r \in F$.

Proof. This follows immediately from the properties of determinant and trace in a matrix. \square

It follows from the above corollary that $\text{Trd}_D : D \rightarrow F$ is an F -module homomorphism. Since all three maps $D \rightarrow D \otimes_F L \rightarrow M_n(L)(d) \xrightarrow{\text{trace}} F$ are graded maps, we have that Trd_D is a graded F -module homomorphism.

Note that a graded division algebra D with centre F is an Azumaya algebra by Theorem 4.3.3. Since the dimension of an Azumaya algebra is a square, it follows that $[D : F]$ is also a square number.

Proposition 6.2.6. *Let D be a graded division algebra over its centre F . Then $\text{Trd}_D : D \rightarrow F$ is surjective.*

Proof. Suppose Trd_D is not surjective. Since $\text{im}(\text{Trd}_D)$ is a graded module over the graded field F with $\dim_F(\text{im}(\text{Trd}_D)) \leq \dim_F(F) = 1$, then $\dim(\text{im}(\text{Trd}_D)) = 0$ and so Trd_D is the zero map. Let $\{x_1, \dots, x_{n^2}\}$ be a homogeneous basis for D over F and let L be a graded maximal subfield of D . Then by Corollary 6.2.4, $f : D \otimes_F L \cong_{\text{gr}} M_n(L)(d)$ for some $(d) \in \Gamma^n$ and it is known that $\{x_1 \otimes 1, \dots, x_{n^2} \otimes 1\}$ forms a homogeneous basis for $D \otimes_F L$ over L . Since f is a graded isomorphism, $\{f(x_i \otimes 1) : 1 \leq i \leq n^2\}$ forms a homogeneous basis of $M_n(L)(d)$ over L . By definition $\text{Trd}_D(d_i) = \text{tr}(f(d_i \otimes 1))$, which equals zero since Trd_D is the zero map. That is, the trace is the zero function on $M_n(L)(d)$, which is clearly a contradiction.

\square

6.3 Some results in the non-graded setting

We recall here some results from the non-graded setting, which will be used in the next section.

Lemma 6.3.1. *Let R be a division ring. If all of the additive commutators of R are central, then R is a field.*

Proof. See [43, Cor. 13.5]. □

Theorem 6.3.2. *Let D be a division algebra over its centre F of index n . Then for $a \in D$, $\text{Trd}_D(a) = na + d_a$ where $d_a \in [D, D]$.*

Proof. Let $a \in D$ with minimal polynomial $f(x) \in F[x]$ of degree m . Then by [56, p. 124, Ex. 9.1], we have

$$f(x)^{n/m} = x^n - \text{Trd}_D(a)x^{n-1} + \cdots + (-1)^n \text{Nrd}_D(a).$$

where the right hand side of this equality is the reduced characteristic polynomial of a . Wedderburn's Factorisation Theorem [42, Thm. 16.9] says $f(x) = (x - d_1 ad_1^{-1}) \cdots (x - d_m ad_m^{-1})$ for $d_1, \dots, d_m \in D$. Combining these, we have

$$\begin{aligned} \text{Trd}_D(a) &= \frac{n}{m} (d_1 ad_1^{-1} + \cdots + d_m ad_m^{-1}) \\ &= \frac{n}{m} (ma + (d_1 ad_1^{-1} - ad_1^{-1} d_1) + \cdots + (d_m ad_m^{-1} - ad_m^{-1} d_m)) \\ &= na + d_a \quad \text{where } d_a \in [D, D], \end{aligned}$$

as required. □

Let R be a commutative Noetherian ring. The dimension of the maximal ideal space of R is defined to be the supremum on the lengths of properly descending chains of irreducible closed sets.

Theorem 6.3.3. *Let R be a commutative Noetherian ring and let A be an Azumaya algebra over R . Then every element of A of reduced trace zero is a sum of at most*

$2d+2$ additive commutators, where d is the dimension of the maximal ideal space of R .

Proof. See [59, Thm. 5.3.1]. □

Corollary 6.3.4. *Let D be a graded division algebra over its centre F , which is Noetherian as a ring. Then $\ker(\text{Trd}_D) = [D, D]$.*

Proof. For any $xy - yx \in [D, D]$, we have $\text{Trd}_D(xy - yx) = 0$ by Corollary 6.2.5. The reverse containment follows immediately from the above theorem, since by Theorem 4.3.3, D is an Azumaya algebra over F . □

Remark 6.3.5. Let D be a graded division algebra over its centre F , which is Noetherian as a ring. Since $\ker(\text{Trd}_D) = [D, D]$ by Corollary 6.3.4 and Trd_D is surjective by Proposition 6.2.6, the First Isomorphism Theorem says that $D/[D, D] \cong_{\text{gr}} F$ as graded F -modules. So $\dim_F(D/[D, D]) = \dim_F F = 1$. By Proposition 4.2.6, $\dim_F([D, D]) + 1 = \dim_F(D) < \infty$.

We recall here the definitions of a totally ordered group and a torsion-free group. Let $(\Gamma, +)$ be a group. A partial order is a binary relation \leq on Γ which is reflexive, antisymmetric and transitive. The order relation is translation invariant if for all $a, b, c \in \Gamma$, $a \leq b$ implies $a + c \leq b + c$ and $c + a \leq c + b$. A partially ordered group is a group Γ equipped with a partial order \leq which is translation invariant. If Γ has a partial order, then two distinct elements $a, b \in \Gamma$ are said to be comparable if $a \leq b$ or $b \leq a$. If Γ is a partially ordered group in which every two elements of Γ are comparable, then Γ is called a *totally ordered group*.

For a group Γ , an element a of Γ is called a torsion element there is a positive integer n such that $a^n = e$. If the only torsion element is the identity element, then the group Γ is called *torsion-free*. By [44], an abelian group can be equipped with a total order if and only if it is torsion-free.

6.4 Quotient division rings

Throughout this section, Γ is a torsion-free abelian group and all graded objects are Γ -graded. In this setting, graded division rings have no zero divisors, and similarly for graded fields. This follows since we can choose a total order for Γ . So for a graded division ring $D = \bigoplus_{\gamma \in \Gamma} D_\gamma$ and two non-zero elements $a, b \in D$, we can write

$$a = a_\gamma + \text{terms of higher degree} \quad \text{and} \quad b = b_\delta + \text{terms of higher degree}.$$

Then $ab = a_\gamma b_\delta + \text{terms of higher degree}$, so we have that ab is non-zero. Thus the group of units of D is $D^* = D^h \setminus 0$. Similarly, a graded field F is an integral domain with group of units $F^* = F^h \setminus 0$. This allows us to construct $QF = (F \setminus 0)^{-1}F$, the *quotient field* of F , which is clearly a field and an F -module. For a graded division algebra D with centre F , we define the *quotient division ring* of D to be $QD = QF \otimes_F D$. We observe some properties of QD in the proposition below, including in part (5) that it is a division ring.

Proposition 6.4.1. *Let D be a graded division algebra with centre F , and let QF and QD be as defined above. Then the following properties hold:*

1. QD is an algebra over QF ;
2. $D \rightarrow QD; d \mapsto 1 \otimes d$ is injective;
3. QD has no zero divisors;
4. $[QD : QF] = [D : F]$;
5. QD is a division ring;
6. $QD \cong (F \setminus 0)^{-1}D$; $(f_1/f_2) \otimes d \mapsto (f_1 d)/f_2$;
7. The elements of QD can be written as d/f , $d \in D, f \in F$.

Proof. (1): This follows easily.

(2): To show $D \rightarrow QD$ is injective, we first show that $F \rightarrow QF$ is injective. Consider $F \rightarrow QF$; $f \mapsto f/1$. If $f/1 = 0$, there is $s \in F \setminus 0$ with $sf = 0$. Since F is an integral domain, it follows that $f = 0$ and the map is injective. As D is graded free over F , and thus flat over F , the required result follows.

(3): Let $x \in QF \otimes D$, say $x = \sum (f_i/f'_i) \otimes d_i$. Then

$$\begin{aligned} x &= (f_1/f'_1) \otimes d_1 + \cdots + (f_k/f'_k) \otimes d_k \\ &= (1/f) \otimes f_1 f'_2 \cdots f'_k d_1 + \cdots + (1/f) \otimes f_k f'_1 \cdots f'_{k-1} d_k \\ &= (1/f) \otimes d, \end{aligned}$$

where $f = f'_1 f'_2 \cdots f'_k$ and $d \in D$. So for any arbitrary element x of $QF \otimes D$, we can write $x = (1/f) \otimes d$, for $f \in F, d \in D$. Now let $x = (1/f) \otimes d$ and $y = (1/f') \otimes d'$ be arbitrary elements of $QF \otimes D$ with $xy = 0$. Then $1/(ff') \otimes dd' = 0$, so

$$(ff' \otimes 1) \left(\frac{1}{ff'} \otimes dd' \right) = 0.$$

Thus $1 \otimes dd' = 0$ and since $D \rightarrow QF \otimes D$ is injective, we have $dd' = 0$. As D has no zero divisors, it follows that $d = 0$ or $d' = 0$; that is, $x = 0$ or $y = 0$ as required.

(4): Since D is free over F and $QD = QF \otimes_F D$, we have

$$[D \otimes_F QF : F \otimes_F QF] = [D : F];$$

that is, $[QD : QF] = [D : F]$.

(5): We will show that any domain which is finite dimensional over a field is a division ring. Since QF is a field, using (3) and (4), it follows that QD is a division ring.

Suppose A is a domain, F is a field, $Z(A) = F$ and $[A : F] = n < \infty$. Let $a \in A \setminus 0$ and consider $1, a, a^2, \dots, a^n$. Then there are $r_i \in F$, not all zero, with $r_0 + r_1 a + \cdots + r_n a^n = 0$. If r_j is the first non-zero element of $\{r_0, \dots, r_n\}$, then

$r_j + r_{j+1}a + \cdots + r_na^{n-j} = 0$. As $a \neq 0$ and F is a field, we have $a(r_{j+1} + \cdots + r_na^{n-j-1})(-r_j)^{-1} = 1$, so a is invertible.

(6): Follows immediately from [48, Prop. 6.55]. Note that they are isomorphic as QF -algebras.

(7): We know from part (3) that if $x \in QD$, then we can write $x = (1/f) \otimes d$. From the isomorphism in (6), $QF \otimes D \cong (F \setminus 0)^{-1}D$; $(1/f) \otimes d \mapsto d/f$. So we can consider the elements of QD as d/f for $d \in D, f \in F \setminus 0$. \square

Note that we have $QF \otimes_F D \cong D \otimes_F QF$ as QF -algebras, so we will use the terms interchangeably. We include here an alternative proof of Lemma 6.1.4, due to Hazrat, which follows from the non-graded result by using the quotient division ring.

Alternative proof of Lemma 6.1.4. Let $y \in D^h$ be an element which commutes with homogeneous additive commutators of D . Then it follows that y commutes with all (non-homogeneous) commutators of D . Consider $[x_1, x_2]$ where $x_1, x_2 \in QD$, with $x_1 = d_1/f_1$ and $x_2 = d_2/f_2$ for $d_1, d_2 \in D, f_1, f_2 \in F$. Then $y[x_1, x_2] = y([d_1, d_2]/f_1f_2) = y[d_1, d_2]/f_1f_2 = [d_1, d_2]y/f_1f_2 = ([d_1, d_2]/f_1f_2)y = [x_1, x_2]y$. So y commutes with all commutators of QD , a division ring. Using Lemma 6.3.1, $y \in QF$.

We will show that $D^h \cap QF \subseteq F^h$. If $x \in D^h \cap QF$, then $x = d/1 = f'/f$ for $d \in D^h, f, f' \in F$ with $f \neq 0$. So there exists $t \in F \setminus 0$ with $tf d = tf'$, so that $d = f^{-1}f'$. Thus $x = (f^{-1}f')/1 \in F$ with $\deg(d) = \deg(f^{-1}f')$; that is, $x \in F^h$. This proves that $y \in F^h$. It follows that all elements of D are commutative, completing the proof. \square

Proposition 6.4.2. *Let D be a graded division algebra over its centre F . Then for $a \in D^h$, the reduced characteristic polynomial of a with respect to D over F coincides with the reduced characteristic polynomial of $a \otimes 1$ with respect to QD over QF .*

Proof. Let L be a splitting field of QD over QF , which exists as QD is a finite dimensional division algebra over QF . So $i : QD \otimes_{QF} L \cong M_n(L)$, and therefore L

is a splitting field of D over F , since

$$j : D \otimes_F L \cong D \otimes_{QF} QF \otimes_F L \cong M_n(L).$$

Then for $a \in D$,

$$\begin{aligned} \text{char}_{QD/QF}(a \otimes 1) &= \det(xI_n - i((a \otimes 1_{QF}) \otimes 1_L)) \\ &= \det(xI_n - j(a \otimes 1_L)) \\ &= \text{char}_{D/F}(a). \end{aligned}$$

So in particular, we have $\text{Trd}_{QD}(a \otimes 1) = \text{Trd}_D(a)$ and $\text{Nrd}_{QD}(a \otimes 1) = \text{Nrd}_D(a)$. \square

Proposition 6.4.3. *Let D be a graded division algebra over its centre F , which is Noetherian as a ring. Then $[D, D] = [QD, QD] \cap D$.*

Proof. We observed in Remark 6.3.5 that $\dim_F([D, D]) + 1 = \dim_F(D) < \infty$. For any $\sum_i x_i y_i - y_i x_i \in [D, D]$, as $D \rightarrow QD$ is injective,

$$1 \otimes \sum_i x_i y_i - y_i x_i = \sum_i (1 \otimes x_i)(1 \otimes y_i) - (1 \otimes y_i)(1 \otimes x_i) \in [QD, QD] \cap D.$$

So we have $[D, D] \subseteq [QD, QD] \cap D \subseteq D$. Here $D \not\subseteq [QD, QD]$, so $[QD, QD] \cap D \neq D$. Thus $[D, D] = [QD, QD] \cap D$. \square

Corollary 6.4.4. *Let D be a graded division algebra with centre F of index n , where F is Noetherian as a ring. Then for each $a \in D$, $\text{Trd}_D(a) = na + d_a$ for some $d_a \in [D, D]$.*

Proof. Let $a \in D$, where $\text{Trd}_{QD}(a \otimes 1) = \text{Trd}_D(a)$ by Proposition 6.4.2. Since QD is a division ring, by Theorem 6.3.2, $\text{Trd}_D(a) = n(a \otimes 1) + c$ where $c \in [QD, QD]$. We know $\text{Trd}_D(a) \in F$. Since $D \rightarrow QD$ is injective, we can consider $n(a \otimes 1)$ as an element of D . So $c = \text{Trd}_D(a) - na \in [QD, QD] \cap D = [D, D]$ by Proposition 6.4.3, as required. \square

The proof of the above corollary shows that using the quotient division ring and the result in the non-graded setting, the graded result follows immediately. We note that this proof only holds for division algebras with a torsion-free grade group.

Corollary 6.4.5. *Let D be a graded division algebra over its centre F , which is Noetherian as a ring. Then*

$$\frac{D}{[D, D]} \otimes_F QF \cong \frac{QD}{[QD, QD]}.$$

Proof. From the above results we have $D/[D, D] \cong_{\text{gr}} F$ as graded F -modules. So

$$\frac{D}{[D, D]} \otimes_F QF \cong QF \cong \frac{QD}{[QD, QD]},$$

where the second isomorphism comes from the non-graded versions of the above results. \square

Remark 6.4.6. Let D be a graded division algebra over its centre F . By definition $\text{SK}_1(D) = D^{(1)}/D'$, where $D^{(1)} = \ker(\text{Nrd}_D)$ and D' is the commutator subgroup of the multiplicative group D^* . We remark that in [34, Thm. 5.7], they have shown that $\text{SK}_1(D) \cong \text{SK}_1(QD)$. Corollary 6.4.5 is similar to this result, where in the above corollary we are considering additive commutators instead of multiplicative ones.

Bibliography

- [1] S. A. Amitsur, *Finite subgroups of division rings*, Trans. Amer. Math. Soc., **80** (1955), 361–386. [5](#)
- [2] M. Auslander, O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc., **97** (1960), 367–409. [36](#)
- [3] G. Azumaya, *On maximally central algebras*, Nagoya Math. J., **2** (1951), 119–150. [7](#), [12](#), [35](#), [36](#)
- [4] H. Bass, *Lectures on topics in algebraic K -theory*, Notes by Amit Roy, Tata Inst. Research Lectures on Math., No. 41, Tata Institute of Fundamental Research, Bombay, 1967. [17](#), [31](#), [36](#)
- [5] M. Boulagouaz, *The graded and tame extensions*, Commutative ring theory (Fès, 1992), 27–40, Lecture Notes in Pure and Appl. Math., **153**, Dekker, New York, 1994. [73](#)
- [6] M. Boulagouaz, *Le gradué d’une algèbre à division valuée*, Comm. Algebra, **23** (1995), no. 11, 4275–4300. [8](#), [63](#)
- [7] N. Bourbaki, *Algebra*, Chapters 1–3, Springer-Verlag, New York, 1989. [16](#), [25](#)
- [8] N. Bourbaki, *Commutative Algebra*, Chapters 1–7, Springer-Verlag, New York, 1989. [61](#)
- [9] A. Braun, *On Artin’s theorem and Azumaya algebras*, J. Algebra, **77** (1982), no. 2, 323–332. [32](#)

-
- [10] A. Braun, *On the defining axioms of Azumaya algebras*, J. Algebra, **109** (1987), no. 1, 166–171. [35](#)
- [11] S. Caenepeel, S. Dăscălescu, C. Năstăsescu, *On gradings of matrix algebras and descent theory*, Comm. Algebra, **30** (2002), no. 12, 5901–5920. [64](#), [89](#)
- [12] S. Caenepeel, F. Van Oystaeyen, Brauer groups and the cohomology of graded rings, Monographs and Textbooks in Pure and Appl. Math., 121, Marcel Dekker, Inc., New York, 1988.
- [13] G. Cortiñas, C. Weibel, *Homology of Azumaya algebras*, Proc. AMS, **121** (1994), no. 1, 53–55. [10](#), [56](#), [62](#)
- [14] E. Dade, *Group-graded rings and modules*, Math. Z., **174** (1980), no. 3, 241–262. [63](#), [105](#)
- [15] S. Dăscălescu, B. Ion, C. Năstăsescu, J. Rios Montes, *Group gradings on full matrix rings*, J. Algebra, **220** (1999), no. 2, 709–728. [92](#), [93](#)
- [16] B. Dawkins, I. Halperin, *The isomorphism of certain continuous rings*, Can. J. Math., **18** (1966), 1333–1344. [54](#)
- [17] F. DeMeyer, E. Ingraham, Separable algebras over commutative rings, Lecture Notes in Mathematics, Vol. 181, Springer-Verlag, Berlin-New York, 1971. [19](#), [21](#), [22](#), [24](#), [30](#), [31](#), [36](#)
- [18] F. R. DeMeyer, G. J. Janusz, *Group rings which are Azumaya algebras*, Trans. Amer. Math. Soc., **279** (1983), no. 1, 389–395. [80](#)
- [19] W. Dicks, *On a characterization of Azumaya algebras*, Publ. Mat., **32** (1988), no. 2, 165–166. [32](#)
- [20] P. K. Draxl, Skew fields, London Mathematical Society Lecture Note Series, 81, Cambridge University Press, Cambridge, 1983. [109](#), [127](#)

-
- [21] S. Endo, Y. Watanabe, *On separable algebras over a commutative ring*, Osaka J. Math., **4** (1967), 233–242. [36](#)
- [22] B. Farb, R. K. Dennis, *Noncommutative algebra*, Graduate Texts in Mathematics, 144. Springer-Verlag, New York, 1993. [17](#), [25](#), [26](#), [28](#)
- [23] R. J. Faudree, *Subgroups of the multiplicative group of a division ring*, Trans. Amer. Math. Soc., **124** (1966), 41–48. [5](#)
- [24] S. Green, D. Handelman, P. Roberts, *K-theory of finite dimensional division algebras*, J. Pure Appl. Algebra, **12** (1978), no. 2, 153–158. [53](#), [54](#), [55](#)
- [25] G. R. Greenfield, *Subnormal subgroups of p-adic division algebras*, J. Algebra, **73** (1981), no. 1, 65–69. [5](#)
- [26] A. Grothendieck, *Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses*, Dix Exposés sur la Cohomologie des Schémas, pp. 46–66, North-Holland, Amsterdam, 1968. [10](#)
- [27] R. Hazrat, *SK₁-like functors for division algebras*, J. Algebra, **239** (2001), no. 2, 573–588. [7](#), [55](#)
- [28] R. Hazrat, *Reduced K-theory of Azumaya algebras*, J. Algebra, **305** (2006), no. 2, 687–703. [7](#), [8](#), [55](#)
- [29] R. Hazrat, R. Hoobler, *K-theory of Azumaya algebras over schemes*, preprint available at: <http://arxiv.org/abs/0911.1406>. [61](#)
- [30] R. Hazrat, M. Mahdavi-Hezavehi, B. Mirzaii, *Reduced K-theory and the group $G(D) = D^*/F^*D'$* , Algebraic K-theory and its applications (Trieste, 1997), 403–409, World Sci. Publ., River Edge, NJ, 1999. [7](#)
- [31] R. Hazrat, J. Millar, *A note on K-Theory of Azumaya algebras*, Comm. Algebra, **38** (2010), no. 3, 919–926. [55](#)

-
- [32] R. Hazrat, J. Millar, *On graded simple algebras*, J. Homotopy Relat. Struct., **5** (2010), no. 1, 113–124. 100
- [33] R. Hazrat, A. R. Wadsworth, *On maximal subgroups of the multiplicative group of a division algebra*, J. Algebra, **322** (2009), 2528–2543. 7, 50
- [34] R. Hazrat, A. Wadsworth, *SK₁ of graded division algebras*, to appear in Israel J. Math, preprint available at: <http://arxiv.org/abs/0812.3433v1>. 8, 9, 137
- [35] I. N. Herstein, *Finite multiplicative subgroups in division rings*, Pacific J. Math., **3** (1953), 121–126. 5
- [36] I. N. Herstein, *Conjugates in division rings*, Proc. AMS, **7** (1956), 10211022. 5
- [37] T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974. 5, 17, 21, 67, 73, 74, 79, 82, 85, 95, 111
- [38] Y.-S. Hwang, A. R. Wadsworth, *Algebraic extensions of graded and valued fields*, Comm. Algebra, **27** (1999), 821–840. 8, 122
- [39] Y.-S. Hwang, A. R. Wadsworth, *Correspondences between valued division algebras and graded division algebras*, J. Algebra, **220** (1999), no. 1, 73–114. 8, 63, 73, 77, 84, 125
- [40] T. Ishikawa, *Faithfully exact functors and their applications to projective modules and injective modules*, Nagoya Math. J., **24** (1964), 29–42. 13
- [41] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Mathematischen Wissenschaften, 294, Springer-Verlag, Berlin, 1991. 17, 19, 21, 25, 31, 36, 129
- [42] T.-Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics, 131. Springer-Verlag, New York, 1991. 5, 49, 119, 131
- [43] T.-Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, 189. Springer-Verlag, New York, 1999. 16, 18, 30, 123, 131

-
- [44] F. W. Levi, *Ordered groups*, Proc. Indian Acad. Sci., Sect. A. **16** (1942), 256–263. [132](#)
- [45] D. W. Lewis, J.-P. Tignol, *Square class groups and Witt rings of central simple algebras*, J. Algebra, **154** (1993), no. 2, 360–376. [7](#), [50](#)
- [46] A. Lichtman, *On subgroups of the multiplicative group of skew fields*, Proc. Amer. Math. Soc., **63** (1977), no. 1, 15–16. [5](#)
- [47] J.-L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften, 301, Springer-Verlag, Berlin, 1998. [56](#), [61](#), [62](#)
- [48] B. Magurn, *An Algebraic Introduction to K -Theory*, Cambridge University Press, 2002. [13](#), [18](#), [19](#), [37](#), [38](#), [39](#), [41](#), [44](#), [60](#), [95](#), [108](#), [112](#), [135](#)
- [49] B. Mitchell, *Theory of categories*, Pure and Applied Mathematics, Vol. XVII Academic Press, New York-London, 1965. [39](#)
- [50] C. Nastasescu, F. Van Oystaeyen, *Dimensions of ring theory*, Mathematics and its Applications, 36, D. Reidel Publishing Co., Dordrecht, 1987. [107](#)
- [51] C. Năstăsescu, F. Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004. [63](#), [68](#), [73](#), [76](#), [81](#), [82](#), [84](#), [93](#), [94](#), [96](#), [101](#), [105](#), [125](#)
- [52] R. S. Pierce, *Associative algebras*, Graduate Texts in Mathematics, 88, Studies in the History of Modern Science, 9, Springer-Verlag, New York-Berlin, 1982. [32](#)
- [53] V. P. Platonov, *The Tannaka-Artin problem and reduced K -theory* (in Russian), Izv. Akad. Nauk SSSR Ser. Mat., **40** (1976), no. 2, 227–261 ; English trans., Math. USSR-Izv., **10** (1976), 211–243. [9](#)
- [54] D. Quillen, *Higher algebraic K -theory I*, Lecture Notes in Mathematics 341, Springer Verlag, Berlin, 1973. [51](#), [52](#)

-
- [55] U. Rehmann, *Zentrale Erweiterungen der speziellen linearen Gruppe eines Schiefkörpers*, J. Reine Angew. Math., **301** (1978), 77–104. 44
- [56] I. Reiner, *Maximal Orders*, Academic Press, New York, 1975. 49, 131
- [57] M. A. Rieffel, *A general Wedderburn theorem*, Proc. Nat. Acad. Sci. U.S.A., **54** (1965), 1513.
- [58] J. Rosenberg, *Algebraic K-Theory and its Applications*, Springer-Verlag, New York, 1994. 6, 37, 38, 40, 41, 44, 51, 60, 100, 102, 103, 104, 112
- [59] M. Rosset, *Elements of trace zero and commutators*, PhD Thesis, Bar-Ilan University, Israel, 1997. 132
- [60] L. H. Rowen, *Azumaya algebras with involution, polarizations, and linear generalized identities*, J. Algebra, **178** (1995), no. 2, 430–443. 35
- [61] L. H. Rowen, *Polynomial identities in ring theory*, Pure and Applied Mathematics, 84, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. 32
- [62] L. H. Rowen, *Ring theory, Vol. II*, Pure and Applied Mathematics, 128, Academic Press, Inc., Boston, MA, 1988. 32
- [63] W. Scharlau, *Quadratic and Hermitian forms*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 270, Springer-Verlag, Berlin, 1985. 25, 32, 49
- [64] J. R. Silvester, *Introduction to algebraic K-theory*, Chapman and Hall Mathematics Series, Chapman & Hall, London-New York, 1981. 37, 42
- [65] G. Szeto, *On separable extensions of group rings and quaternion rings*, Internat. J. Math. Math. Sci., **1** (1978), no. 4, 433–438. 32
- [66] J.-P. Tignol, A. R. Wadsworth, *Totally ramified valuations on finite-dimensional division algebras*, Trans. Amer. Math. Soc., **302** (1987), no. 1, 223–250. 8

-
- [67] J.-P. Tignol, A. R. Wadsworth, *Value functions and associated graded rings for semisimple algebras*, to appear in Trans. Amer. Math. Soc. **8**
- [68] L. N. Vaserstein, *On the Whitehead determinant for semi-local rings*, J. Algebra, **283** (2005), no. 2, 690–699. **43**
- [69] O. E. Villamayor, D. Zelinsky, *Galois theory for rings with finitely many idempotents*, Nagoya Math. J., **27** (1966), 721–731. **22**
- [70] C. Weibel, An introduction to algebraic K-theory, available online at: <http://www.math.rutgers.edu/~weibel/Kbook.html> **37, 39, 42, 44, 51, 52**
- [71] C. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994. **24, 56, 61**
- [72] C. T. C. Wall, *Graded Brauer groups*, J. Reine Angew. Math., **213** (1963/1964), 187–199. **63, 77**

Index

- abelianisation, 4
- additive commutator, 115
- algebra
 - Azumaya, 23, 29
 - central, 27
 - classically separable, 22
 - enveloping, 17
 - étale, 27
 - finitely presented, 27
 - separable, 17
- anti-homomorphism, 32
- Azumaya algebra, 23, 29
- central algebra, 27
- classically separable algebra, 22
- commutator subgroup, 4
- constant rank, 57
- crossed product, 62
- \mathcal{D} -functor, 53
- Dieudonné determinant, 40
- elementary matrix, 38
- enveloping algebra, 17
- étale algebra, 27
- exact functor, 11
- faithful functor, 28
- faithful module, 12
- faithfully exact functor, 11
- faithfully flat module, 14
- faithfully projective module, 14
- field extension
 - separable, 22
- finitely presented algebra, 27
- flat module, 12
- functor
 - exact, 11
 - faithful, 28
 - faithfully exact, 11, 12
 - full, 28
 - fully faithful, 28
- general linear group, 38
- generator module, 28
- good grading, 88
- graded algebra, 72
 - graded central simple, 73
- graded Azumaya algebra, 91
- graded central simple algebra, 73
- graded \mathcal{D} -functor, 107
- graded division algebra, 115

-
- graded division ring, 69
 - graded field, 73
 - graded ideal, 63
 - graded matrix ring, 77
 - graded module, 65
 - graded faithfully projective, 89
 - graded free, 65
 - graded projective, 89
 - graded simple, 65
 - graded module homomorphism, 65
 - graded Morita equivalence, 92
 - graded ring, 61
 - graded simple, 63
 - graded ring homomorphism, 64
 - graded submodule, 65
 - group
 - torsion-free, 126
 - totally ordered, 126
 - Hochschild homology module, 59
 - homogeneous additive commutator, 115
 - homogeneous element, 61
 - homogeneous ideal, 63
 - involution, 32
 - Jacobson radical, 22
 - local ring, 57
 - module
 - faithful, 12
 - faithfully flat, 14
 - faithfully projective, 14
 - flat, 12
 - generator, 28
 - projective, 12
 - projective generator, 28
 - Morita equivalence, 37
 - principal ideal domain, 38
 - progenerator module, 28
 - projective generator module, 28
 - projective module, 12
 - proper maximally central, 33
 - quotient division ring, 127
 - quotient field, 127
 - ring
 - local, 57
 - semi-local, 40
 - semi-simple, 37
 - semi-local ring, 40
 - semi-simple ring, 37
 - separability idempotent, 18
 - separable algebra, 17
 - separable field extension, 22
 - Steinberg group, 41
 - strongly graded ring, 62
 - support, 61
 - torsion group, 44

torsion-free group, 126

totally ordered group, 126

totally ramified, 115

trace ideal, 28

unramified, 115